Lognormal Stochastic Volatility with Applications to Cryptocurrency Options

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References:

• Sepp A and Rakhmonov P (2023) Log-normal Stochastic Volatility Model with Quadratic Drift, International Journal of Theoretical and Applied Finance, 2023, 26(8)
`https://www.worldscientific.com/doi/reader/10.1142/S0219024924500031`

• Lucic V and Sepp A (2024) Valuation and Hedging of Cryptocurrency Inverse Options, Quantitative Finance, 2024, 24(7), 851-869
`https://ssrn.com/abstract=4606748`

Applications of stochastic volatility models

1. Inference of volatility from discrete time series
   - Open-High-Low-Close estimators, GARCH type models
   - Focus on prediction ability of future volatility

2. Vanilla options valuation using dynamic model for returns volatility: modelling vanilla options and estimation of risk-premia for systematic strategies
   - Heston model
   - Exp-OU, Bergomi type models
   - Log-normal model
   - Focus on prediction ability of implied volatility dynamics

3. Structured products valuation and risk management: applications of models for managing derivatives books
   - Dupire local volatility models
   - Local stochastic volatility models
   - Focus on close fit to traded vanilla options and risk decomposition of structured products

Heston model

- Heston model dynamics for price S_t and variance V_t:

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sqrt{V_t} S_t dW_t, \quad S_0 = S, \\
dV_t &= \kappa\,(\theta - V_t)\,dt + \varepsilon\sqrt{V_t} dZ_t, \quad V_0 = V,
\end{aligned}
\tag{1}
$$

where μ_t is price drift under $\mathbb{P}$ and $\mu_t = r_t$ under risk-neutral measure $\mathbb{Q}$
W_t and Z_t are Brownian motions with return-volatility correlation ρ
κ is mean-reversion speed
θ is long-run variance
ε is volatility of volatility

- Drawbacks:

1) Heston model provides semi-analytic solution for vanilla options but it is hard to implement numerically for Monte-Carlo and PDE solvers

2) The variance can reach zero is reachable if Feller condition is not satisfied

Exp-OU, Bergomi type models

• Dynamics expressed using the forward variance $\xi_t(u)$

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sqrt{\xi_t(t)} S_t dW_t, \quad S_0 = S, \\
\xi_t(u) &= \xi_0(u) \exp\left\{\varepsilon e^{-\kappa(u-t)} Y_t - \frac{1}{2}\varepsilon^2 e^{-2\kappa(u-t)} \mathbb{E}[Y_t^2]\right\} \qquad (2) \\
dY_t &= -\kappa Y_t dt + dZ_t
\end{aligned}
$$

where W_t and Z_t are Brownians with return-volatility correlation ρ
Y_t is Ornstein–Uhlenbeck (OU) driver

• Exp-OU, Bergomi type models provide no analytic solution for vanilla but they are easier to implement numerically

• Drawbacks:

1) These models cannot be applied for assets with positive volatility-return correlations because the asset price loses martingale property

2) The model dynamics are not closed (the functional form of the volatility process is different) under the change of numeraire so that they cannot be applied for valuation of inverse options using spot price numeraire

Log-normal stochastic volatility model

• The dynamics for price S_t and volatility σ_t:

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t^{(0)}, \\ d\sigma_t &= (\kappa_1 + \kappa_2 \sigma_t)(\theta - \sigma_t)dt + \beta \sigma_t dW_t^{(0)} + \varepsilon \sigma_t dW_t^{(1)}, \end{aligned} \tag{3}$$

$W_t^{(0)}$ and $W_t^{(1)}$ are uncorrelated Brownian motions
$\beta \in \mathbb{R}$ is the volatility beta to price changes
$\varepsilon \in \mathbb{R}_+$ is residual volatility-of-volatility
$\theta \in \mathbb{R}_+$ is the mean level of the volatility
$\kappa_1 + \kappa_2 \sigma_t$ is the mean-reversion speed is linear function of the volatility

• The model combines the best of the two worlds: semi-analytic solution for vanilla options and easy to implement numerically

1) The model can be applied for assets with positive return-volatility correlation (if $\kappa_2 > \max\{0, \beta\}$)

2) The model is closed under change of numeraire so that can be applied for valuation of inverse options

• Classic results do not apply due to super-linear drift in SDE for σ_t

Theorem 1. i) *Boundaries* $\{0, +\infty\}$ *are unattainable for volatility* σ_t
ii) *SDE for* σ_t *has a unique strong solution.*

Model Dynamics consistent with Forward Variance I

• Given a set of put and call option prices, we compute fair strikes of variance swaps

$$K^2_{varswap}(T) = \frac{2}{T}\left(\sum_{K<F(T)} \frac{\delta K}{K^2} Put(T,K) + \sum_{K\geq F(T)} \frac{\delta K}{K^2} Call(T,K)\right)$$

• Exp-OU Bergomi model type are formulated as forward variance models under which: $\mathbb{E}[d\sigma_t^2] = \delta K^2_{varswap}(t)dt$

• The model is fitted by construction to the term structure of market-implied var swaps for any set of model parameters

• Formulate log-normal SV model using volatility backbone function $\eta(t)$

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t \eta(t) S_t dW_t^{(0)},\ S_0 = S, \\ d\sigma_t &= (\kappa_1 + \kappa_2 \sigma_t)(\theta - \sigma_t)dt + \beta\sigma_t dW_t^{(0)} + \varepsilon\sigma_t dW_t^{(1)},\ \sigma_0 = \sigma, \\ dI_t^{model} &= \eta^2(t)\sigma_t^2 dt,\ I_0^{model} = 0, \\ dI_t &= \sigma_t^2 dt,\ I_0 = 0, \end{aligned} \tag{4}$$

Model Dynamics consistent with Forward Variance II

- Consider $\eta(t)$ to be a deterministic piecewise function

$$\eta(t) = \sum_n \eta_n \mathbb{1}_{\{t \in (T_{n-1}, T_n]\}}$$

- Model-implied value of expected quadratic variance

$$\bar{I}_t^{model}(T) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \eta^2(t')\sigma_{t'}^2 dt' \mid \mathcal{F}_t\right] = \sum_n \eta_n^2 \left(\mathbb{E}^{\mathbb{Q}}\left[I_{T_n}\right] - \mathbb{E}^{\mathbb{Q}}\left[I_{T_{n-1}}\right]\right) \quad (5)$$

- We apply the analytic formula for computing expected value of quadratic variance $\bar{I}(T) = \mathbb{E}^{\mathbb{Q}}[I_T]$ (initial value σ_0 and mean θ are fixed externally)

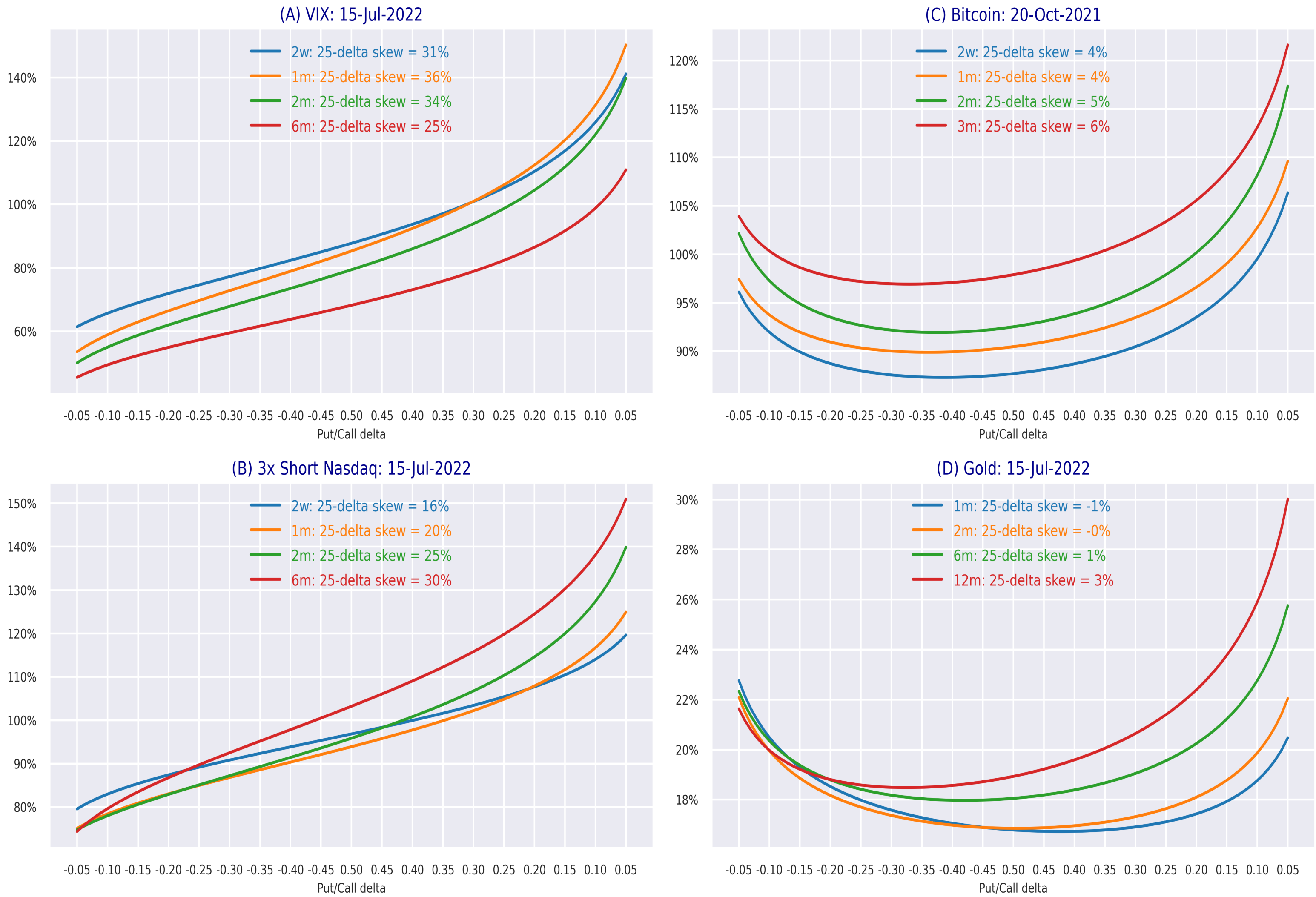
- The term structure of η_n is fitted as follows:

$$\eta_n^2 = \frac{T_n K_{varswap}^2(T_n) - T_{n-1} K_{varswap}^2(T_{n-1})}{\bar{I}(T_n) - \bar{I}(T_{n-1})}$$

Assets with positive volatility return correlations

Many assets have positive implied Black-Scholes volatility skews

Volatilities of out-of-the-money calls are higher than volatilities of out-of-the-money puts



Vanilla and inverse options

- **Pay-offs of vanilla call and put** are settled in cash at expiry time T:

$$u^{\text{call}}(S_T) = \max\{S_T - K, 0\}\,,\;\; u^{\text{put}}(S_T) = \max\{K - S_T, 0\}\,.$$

- **Pay-offs of inverse call and put** are converted to units of asset at T:

$$\tilde{u}^{\text{call}}(S_T) = \frac{1}{S_T}\max\{S_T - K, 0\}\,,\;\; \tilde{u}^{\text{put}}(S_T) = \frac{1}{S_T}\max\{K - S_T, 0\}\,.$$

- In markets for cryptocurrency options, inverse payoffs are popular for coin-margined accounts and on-chain transfers (Deribit exchange)

- Vanilla options on cryptocurrencies are also traded (Binance, CME exchange)

- Are there arbitrage opportunities between inverse and vanilla options?

Valuation under MMA and inverse measures

• By choosing money market account (MMA) $M(T)$, $M(T) = \exp\left\{\int_t^T r(s)ds\right\}$, as a numéraire, we consider an equivalent martingale measure $\mathbb{Q}$, induced by M, where $r(t)$ is risk-free rate.

• Accordingly, the time-t value of an option, denoted by $U(t,S)$, with payoff function $u(S_T)$ at time T, equals to

$$U(t,S) = M(t)\mathbb{E}\left[\frac{1}{M(T)}\, u(S_T)|\,\mathcal{F}_t\right], \tag{6}$$

where expectation $\mathbb{E}$ is taken under the MMA martingale measure $\mathbb{Q}$.

• By choosing S as a numéraire, we consider an inverse martingale measure $\tilde{\mathbb{Q}}$, induced by S.

• The value function of inverse option, denoted by $\tilde{U}(t,S)$, with the payoff paid in units of S, equals to

$$\tilde{U}(t,S) = S_t\tilde{\mathbb{E}}\left[\frac{1}{S_T}\, u(S_T)|\,\mathcal{F}_t\right], \tag{7}$$

where expectation $\tilde{\mathbb{E}}$ is taken under the inverse martingale measure $\tilde{\mathbb{Q}}$.

Equivalence of MMA and inverse measures

• Are values of the vanilla option under the MMA measure and of inverse option under the inverse measure equal?

Theorem 2. *Assume a complete market and that both the MMA measure $\mathbb{Q}$ and the inverse measure $\tilde{\mathbb{Q}}$ are equivalent martingale measures. Then the values of options under the MMA measure in Eq (6) and under the inverse measure are in Eq (7) satisfy*

$$M(t)\mathbb{E}\left[\frac{1}{M(T)}\,u(S_T)\middle|\,\mathcal{F}_t\right] = S_t\tilde{\mathbb{E}}\left[\frac{1}{S_T}u(S_T)\middle|\mathcal{F}_t\right]. \tag{8}$$

Proof. Using the change of numéraire technique in theorem 1 in *Geman et al. (1995)*. □

• PV of inverse option = PV of vanilla option / Price

• Dynamic SV models must produce martingale dynamics for both MMA and inverse measures

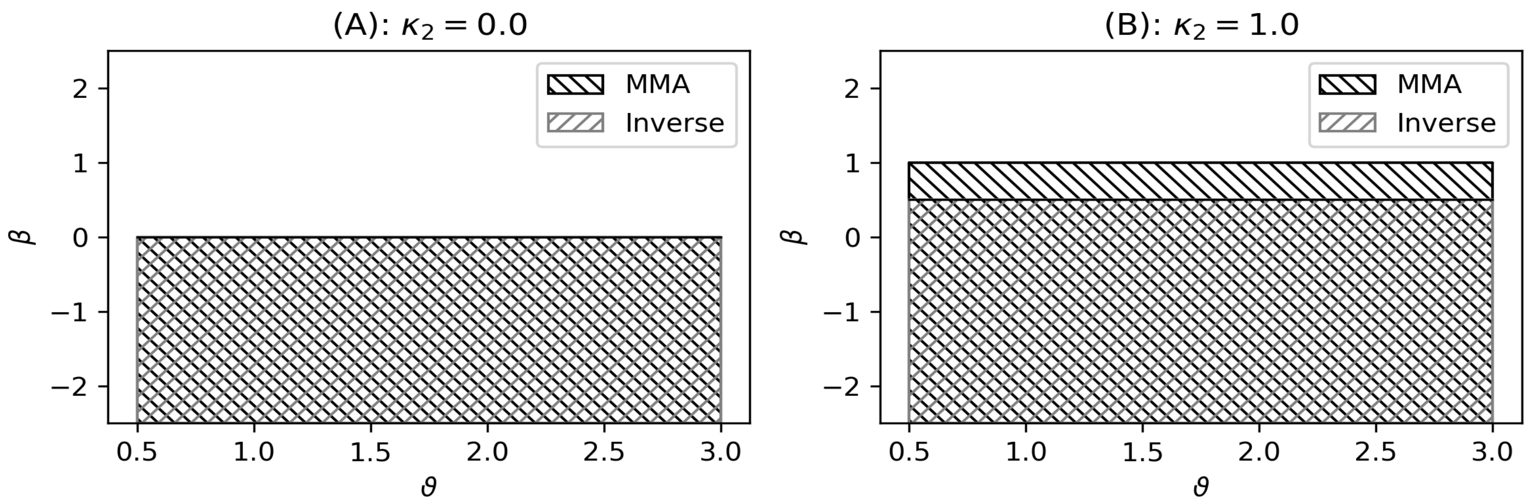
Importance of quadratic drift for Log-normal SV model

Theorem 3. *[Log-normal SV model with quadratic drift] Under model dynamics (3) for the price processes S_t and $R_t = S_t^{-1}$.*
1) *The process S_t is a martingale under the MMA measure $\mathbb{Q}$ iff $\kappa_2 \geq \beta$.*

2) *The process R_t is a martingale under the inverse measure $\tilde{\mathbb{Q}}$ iff $\kappa_2 \geq 2\beta$*

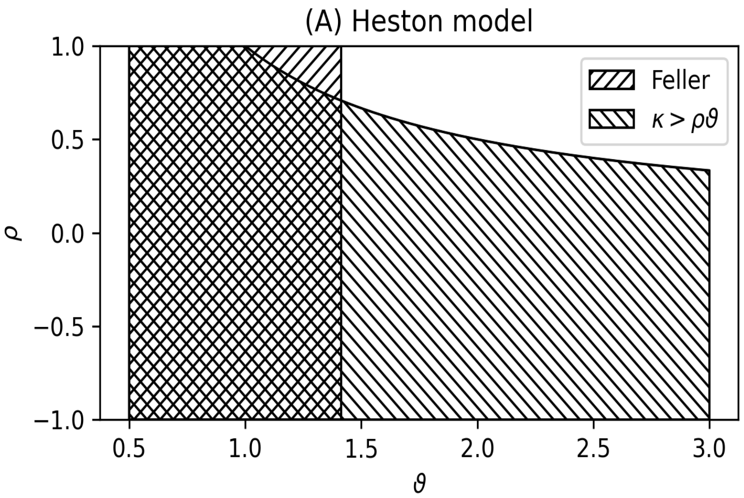
• Linear Log-normal SV model with $\kappa_2 = 0$ is not arbitrage-free for $\beta > 0$like classic Exp-OU SV models

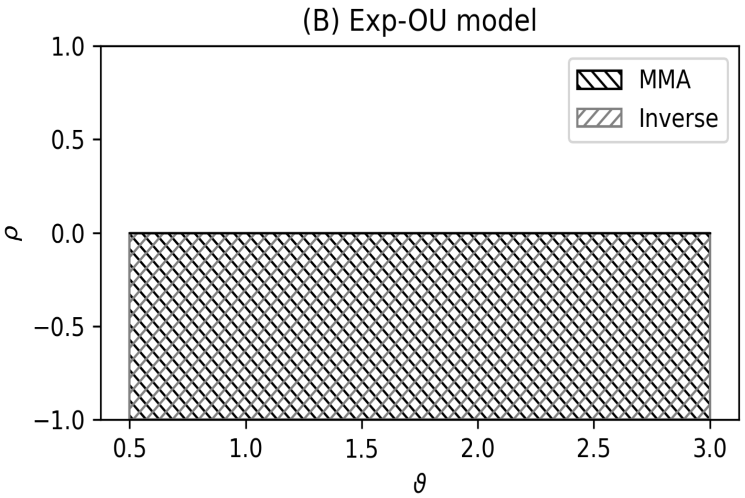
• Figure: Admissible regions of model parameters (β, ϑ) for the martingale property of Z_T under the MMA measure $\mathbb{Q}$ and of R_T under the inverse measure $\tilde{\mathbb{Q}}$ with $\kappa_2 = 0$ and $\kappa_2 = 1$.
• Both conditions are satisfied in the area with overlapping hatches.



Conventional SV models are ill-defined when return-volatility correlation is positive

- Heston model may not have a stationary distribution of the variance when $\kappa - \rho\vartheta < 0$
- Exp-OU SV model is not arbitrage-free for positive return-vol correlation
- Subplot (A) shows the admissible region for parameters (ρ, ϑ) of Heston model where Feller condition is satisfied and the stationary distribution exists under the inverse measure using $\kappa = 1$, $\theta = 1$
- Subplot (B) shows the admissible region parameters (ρ, ϑ) of Exp-OU SV model for the martingale property under the MMA and inverse measures using $\kappa = 1$, $\theta = 1$
- Both conditions are satisfied in the area with overlapping hatches





Properties of the log-normal volatility process

• We consider the SDE of volatility process in model dynamics in Eq (3) represented as follows

$$d\sigma_t = (\kappa_1 + \kappa_2\sigma_t)(\theta - \sigma_t)dt + \vartheta\sigma_t dW_t^{(*)}, \ \sigma_0 = \sigma \tag{9}$$

where $W_t^{(*)}$ is a standard Brownian motion
$\vartheta^2 = \beta^2 + \varepsilon^2$ is the total variance of volatility.

• Notice that the process σ_t is not a polynomial diffusion because the drift coefficient is a second-order polynomial in σ_t (using Lemma 2.2 in *Filipovic and Larsson (2016)*)

Lemma 1. *Dynamics of volatility in Eq (3) under the inverse measure* $\tilde{\mathbb{Q}}$

$$d\sigma_t = \left(\kappa_1\theta - (\kappa_1 - \kappa_2\theta)\sigma_t - (\kappa_2 - \beta)\sigma_t^2\right)dt + \beta\sigma_t d\widetilde{W}_t^{(0)} + \varepsilon\sigma_t d\widetilde{W}_t^{(1)}.$$

Using zero-drift stochastic driver Z_t, $Z_t = e^{-\int_0^t r(t')dt'} S_t$, *the MMA measure* $\mathbb{Q}$ *and the inverse measure* $\tilde{\mathbb{Q}}$ *are related by the density process by*

$$\Lambda_t = \mathbb{E}_t\left[d\tilde{\mathbb{Q}}/d\mathbb{Q}\right] = Z_t/Z_0.$$

Theorem 4. *Measures* $\mathbb{Q}$ *and* $\tilde{\mathbb{Q}}$ *are equivalent if and only if* $\kappa_2 \geq \beta$.

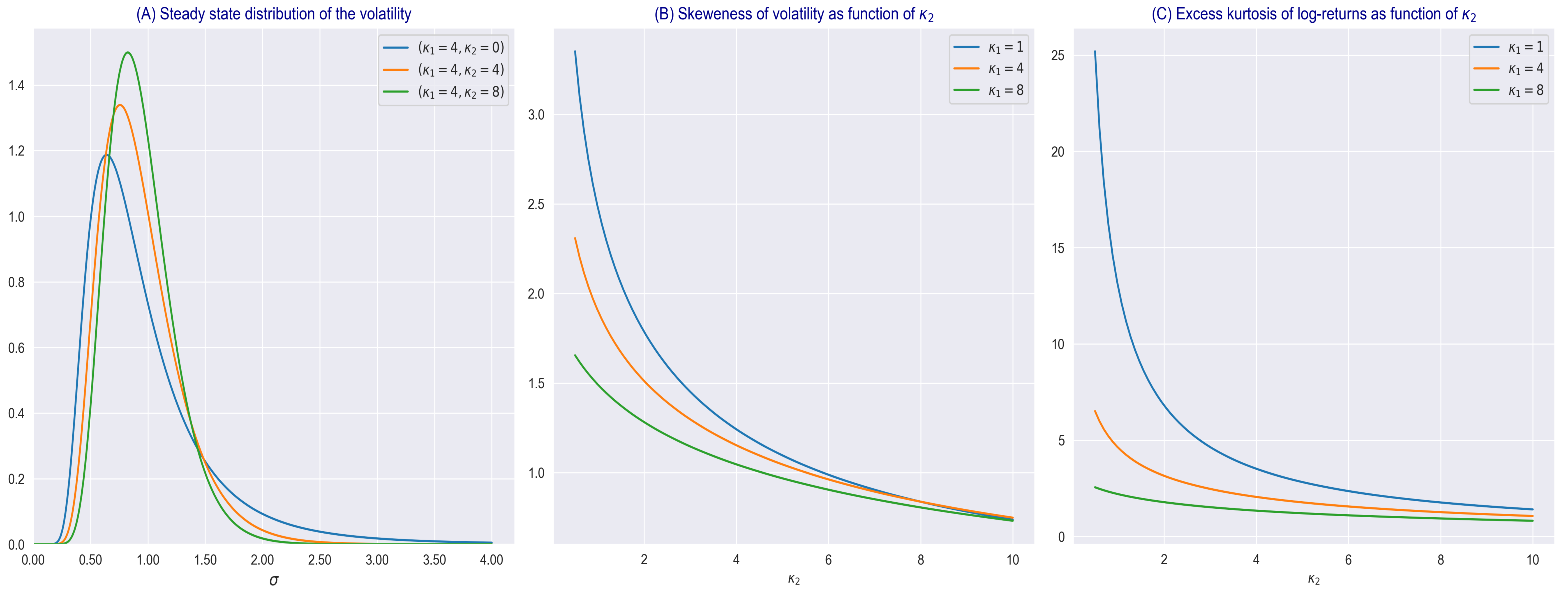
Steady state distribution of the volatility

• The steady state distribution is given by the Generalized Inverse Gaussian distribution (*Jorgensen (1982)*)

$$G(\sigma) = c\sigma^{\eta-1} \exp\left\{-\left(\frac{q}{\sigma} + b\sigma\right)\right\}, \qquad \sigma > 0 \tag{10}$$

• Subplot (A) shows the steady state PDF of the volatility computed using Eq (10) with fixed $\kappa_1 = 4$ and $\kappa_2 = \{0, 4, 8\}$.

• Subplot (B) and (C) show the skeweness of the volatility and the excess kurtosis of the unconditional returns distribution , respectively, both as functions of κ_2, for the three choices of $\kappa_1 = 1, 4, 8$.
Other model parameters are fixed to $\theta = 1$ and $\vartheta = 1.5$.



Moments of Volatility Process, I

• We consider the mean-adjusted process $Y_t = \sigma_t - \theta$ and define its power function $m_t^{(n)}$, $m_t^{(n)} = Y_t^n$, and the m-th moment, $\overline{m}_\tau^{(n)}$, $n = 0, 1, 2, ..,$ as follows

$$\overline{m}_t^{(n)}(\tau) = \mathbb{E}_t\left[m_\tau^{(n)}\right]$$

• By applying Itô's lemma for $m_t^{(n)}$, we obtain

$$dm_t^{(n)} = \left(-\kappa Y_t - \kappa_2 Y_t^2\right) nY^{n-1}dt + c(n)\left(Y_t + \theta\right)^2 Y^{n-2}dt + \vartheta\left(Y_t + \theta\right) nY^{n-1}dW_t^{(}$$

with $m_0^{(n)} = Y_0^n$ and $c(n) = \frac{1}{2}\vartheta^2 n(n-1)$

• We notice a pattern of the powers of n which allows for a recursive solution as follows.

Moments of Volatility Process, II

Proposition 1. *[Moments of Volatility Process] The solution to moments, can be presented as a matrix equation for an infinite-dimensional vector:*

$$\partial_\tau M^{(0,\infty)}(\tau) = \Lambda^{(0,\infty)} M^{(0,\infty)}(\tau),$$

where ∂_τ *is the derivative w.r.t.* τ *and*

$$M^{(0,\infty)}(\tau) = \left(\overline{m}^{(0)}, \overline{m}^{(1)}, \overline{m}^{(2)}, \overline{m}^{(3)}, \overline{m}^{(4)}, ...\right)^T, \ M^{(0,\infty)}(0) = \left(1, Y_0, Y_0^2, Y_0^3, Y_0\right.$$

$$\Lambda^{(0,\infty)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & ... \\ 0 & -\kappa & -\kappa_2 & 0 & 0 & 0 & ... \\ c(2)\theta^2 & 2c(2)\theta & (c(2)-2\kappa) & -2\kappa_2 & 0 & 0 & ... \\ 0 & c(3)\theta^2 & 2c(3)\theta & (c(3)-3\kappa) & -3\kappa_2 & 0 & ... \\ 0 & 0 & c(4)\theta^2 & 2c(4)\theta & (c(4)-4\kappa) & -4\kappa_2 & ... \\ ... & & & & & & \end{pmatrix}.$$

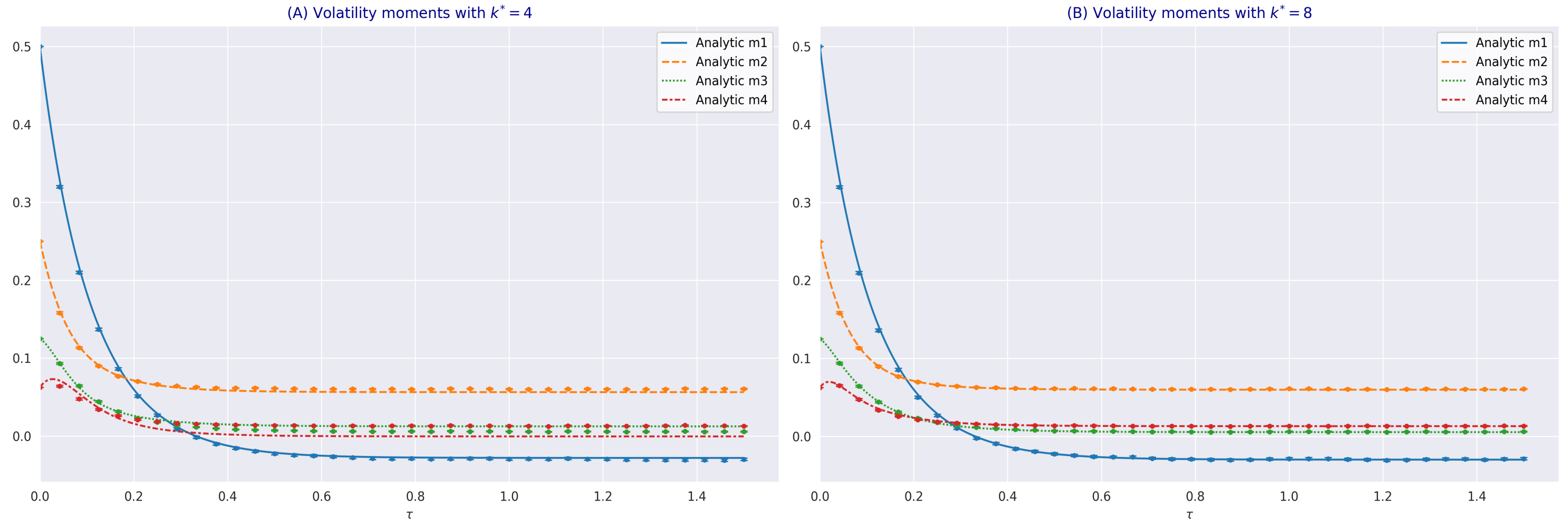
Moments of Volatility Process, III

• An approximate solution to ODE system is obtained using a truncation by fixing the number of terms to k^* and using a finite dimensional vector of m-th moments, $m = 1, ..., k^*$ with analytic solution

$$M^{(1,k^*)}(\tau) = \text{expm}\left\{\Lambda^{(1,k^*)}t\right\} \cdot M^{(1,k^*)}(0) + \left(\Lambda^{(1,k^*)}\right)^{-1} \cdot \left(\text{expm}\left\{\Lambda^{(1,k^*)}t\right\} - I^{(k^*)}\right)$$

where expm() is the matrix exponent, $\cdot$ and $^{-1}$ are the matrix product and inverse, respectively, and $I^{(k^*)}$ is $k^* \times k^*$ identity matrix.

• Subplot (A) and (B): The first four moments of mean-adjusted volatility computed with the truncation order $k^* = 4$ and $k^* = 8$ as functions of τ. Dots and error bars denote the estimate and 95% confidence interval, respectively, computed using MC simulations of model dynamics.



Model Dynamics under MMA and Inverse Measures

We introduce the mean-adjusted volatility process $Y_t = \sigma_t - \theta$

Corollary 1 (Dynamics under the MMA measure $\mathbb{Q}$). *The joint dynamics of log-price $X_t = \log S_t$, the mean-adjusted volatility process $Y_t = \sigma_t - \theta$, and the QV I_t:*

$$dX_t = -\frac{1}{2}\eta^2(t)\,(Y_t+\theta)^2\,dt + \eta(t)\,(Y_t+\theta)\,dW_t^{(0)},\ X_0 = X,$$
$$dY_t = \left(-\kappa Y_t - \kappa_2 Y_t^2\right)dt + \beta\,(Y_t+\theta)\,dW_t^{(0)} + \varepsilon\,(Y_t+\theta)\,dW_t^{(1)},\ Y_0 = \sigma_0 - \theta,$$
$$dI_t = \eta^2(t)\,(Y_t+\theta)^2\,dt,\ I_0 = I.$$

Corollary 2 (Dynamics under the inverse measure $\tilde{\mathbb{Q}}$). *The joint dynamics of log-price X_t, the mean-adjusted volatility process Y_t, and the QV I_t:*

$$dX_t = \frac{1}{2}\eta^2(t)\,(Y_t+\theta)^2\,dt + (Y_t+\theta)\,\eta(t)d\widetilde{W}_t^{(0)},\ X_0 = X,$$
$$dY_t = \left(\tilde{\lambda} - \tilde{\kappa} Y_t - \tilde{\kappa}_2 Y_t^2\right)dt + \beta\,(Y_t+\theta)\,d\widetilde{W}_t^{(0)} + \varepsilon\,(Y_t+\theta)\,d\widetilde{W}_t^{(1)},\ Y_0 = \sigma_0 - \theta,$$
$$dI_t = \eta^2(t)\,(Y_t+\theta)^2\,dt,\ I_0 = I.$$

where $\tilde{\lambda} = \beta\theta^2\eta(t),\ \tilde{\kappa} = \kappa_1 - \kappa_2\theta + 2(\kappa_2 - \beta\eta(t))\theta,\ \tilde{\kappa}_2 = \kappa_2 - \beta\eta(t)$

Joint Valuation Equation under MMA and Inverse measures

• State variables: $X_t = \ln S_t$, $Y_t = \sigma_t - \theta$, $I_t = \int_0^t \sigma_t^2 dt$

The joint valuation PDE under MMA $p = 1$ and inverse $p = -1$ measures:

$$U(\tau, X, I, Y; p) = \begin{cases} \mathbb{E}^{\mathbb{Q}}[u(X_T) \mid \mathcal{F}_t], & p = 1 \\ \tilde{\mathbb{E}}^{\tilde{\mathbb{Q}}}[u(X_T) \mid \mathcal{F}_t], & p = -1. \end{cases} \tag{11}$$

• The classic Feynman-Kac formula cannot be applied directly to Eq (11) because the model dynamics do not satisfy linear growth condition

Theorem 5. *The value function $U(\tau, X, Y; p)$ solves the PDE:*

$$-U_\tau + \left(\mathcal{L}^{(Y;p)} + \mathcal{L}^{(X;p)} + \mathcal{L}^{(I;p)}\right) U = 0,\ U(0, X, I, Y) = u(X, I), \tag{12}$$

with operators:

$$\mathcal{L}^{(Y;p)}U = \frac{1}{2}\vartheta^2 (Y+\theta)^2 U_{YY} + \left(\lambda^{(p)} - \kappa^{(p)} Y - \kappa_2^{(p)} Y^2\right) U_Y,$$

$$\mathcal{L}^{(X;p)}U = (Y+\theta)^2 \left[\frac{1}{2}\eta^2(t)\left(U_{XX} - pU_X\right) + \beta\eta(t)U_{XY}\right],\ \mathcal{L}^{(I;p)}U = (Y+\theta)^2 U$$

$$\kappa^{(p)} = \kappa_1 - \kappa_2\theta + 2\kappa_2^{(p)}\theta,\ \lambda^{(p)} = (\kappa_2 - \kappa_2^{(p)})\theta^2,\ \kappa_2^{(p)} = \begin{cases} \kappa_2, & p = 1, \\ \kappa_2 - \beta\eta(t), & p = -1 \end{cases}$$

Moment Generating Function (MGF) for state variables

• Introduce MGF for state variables (X_t, I_t, Y_t) with complex-valued transform parameters $\Phi, \Psi, \Theta \in \mathbb{C}$:

$$G(\tau, \Phi, \Psi, \Theta; p) = \begin{cases} \mathbb{E}^{\mathbb{Q}}[e^{-\Phi X_\tau - \Psi I_\tau - \Theta Y_\tau} \,|\, \mathcal{F}_t], & p = 1 \\ \tilde{\mathbb{E}}^{\tilde{\mathbb{Q}}}[e^{-\Phi X_\tau - \Psi I_\tau - \Theta Y_\tau} \,|\, \mathcal{F}_t], & p = -1 \end{cases} \tag{13}$$

• G solves the PDE:

$$\begin{aligned} & -G_\tau + \left(\mathcal{L}^{(Y;\,p)} + \mathcal{L}^{(X;\,p)} + \mathcal{L}^{(I;\,p)}\right) G = 0, \\ & G(0, \Phi, \Psi, \Theta; p) = e^{-\Phi X - \Psi I - \Theta Y} \end{aligned} \tag{14}$$

• MGF in (14) has no closed-form solution because volatility generator $\mathcal{L}^{(Y;\,p)}$ is not affine in both variance and drift terms:

$$\mathcal{L}^{(Y;\,p)} U = \frac{1}{2}\vartheta^2 (Y + \theta)^2 U_{YY} + \left(\lambda^{(p)} - \kappa^{(p)} Y - \kappa_2^{(p)} Y^2\right) U_Y$$

• Analytical intractability has limited applications of log-normal SV models despite their empirical support

Existence of MGF for state variables

Theorem 6. *Given the transform variable $\Phi = \Phi_R + i\Phi_I \in \mathbb{C}$, the MGF of the log-price X_τ*

$$G(\tau, X; \Phi; p = 1) = \mathbb{E}[e^{-\Phi X_\tau}\Big| X_0 = X]$$

exists for $\Phi_R \in (-1, 0)$, if Z_τ is a martingale under the MMA measure. By Theorem 3 1), the necessary condition is $\kappa_2 \geq \beta$.

Similarly, the MGF of the log-price X_τ

$$G(\tau, X; \Phi; p = -1) = \tilde{\mathbb{E}}[e^{-\Phi X_\tau}\Big| X_0 = X]$$

exists for $\Phi_R \in (0, 1)$ if R_τ is a martingale under inverse measure. By Theorem 3 2), the necessary condition is $\kappa_2 \geq 2\beta$.

Affine expansion for closed-form solution

• Make an anzats for exponential solution $E^{[\infty]}$ using an infinite dimensional vector $\mathbf{A}(\tau) = \left\{A^{(k)}(\tau; \Phi, \Psi, \Theta;\, p)\right\},\ k = 0, 1, ..., \infty$:

$$E^{[\infty]}(\tau, \Phi, \Psi, \Theta;\, p) = \exp\left\{-\Phi X - \Psi I + \sum_{k=0}^{\infty} A^{(k)}(\tau; \Phi, \Psi, \Theta;\, p) Y^k\right\},$$

where $\mathbf{A}(\tau)$ solves an infinite-dimensional system of quadratic ODEs:

$$A_\tau^{(k)} = \mathbf{A}^\top M^{(k)} \mathbf{A} + \left(L^{(k)}\right)^\top \mathbf{A} + H^{(k)},\ \ k = 0, 1, ..., \infty, \tag{15}$$

with $M^{(k)}$, $L^{(k)}$ and $H^{(k)}$ infinite dimensional (very) sparse matrices

• Similarity with the problem of computing the moments of the volatility process: we need to introduce a finite dimensional basis

Theorem 7. *[Second-order affine expansion] The MGF in Eq* (13) *can be decomposed into leading term $E^{[4]}$ and remainder term $R^{[4]}$:*

$$G(\tau, \Phi, \Psi, \Theta;\, p) = E^{[4]}(\tau, \Phi, \Psi, \Theta;\, p) + R^{[4]}(\tau, \Phi, \Psi, \Theta;\, p), \tag{16}$$

Theorem 8. *Continuous solution $\mathbf{A}$ for $E^{[4]}$ exists on* $[0, \tau_0)$

Approximation for MGF using second-order affine expansion

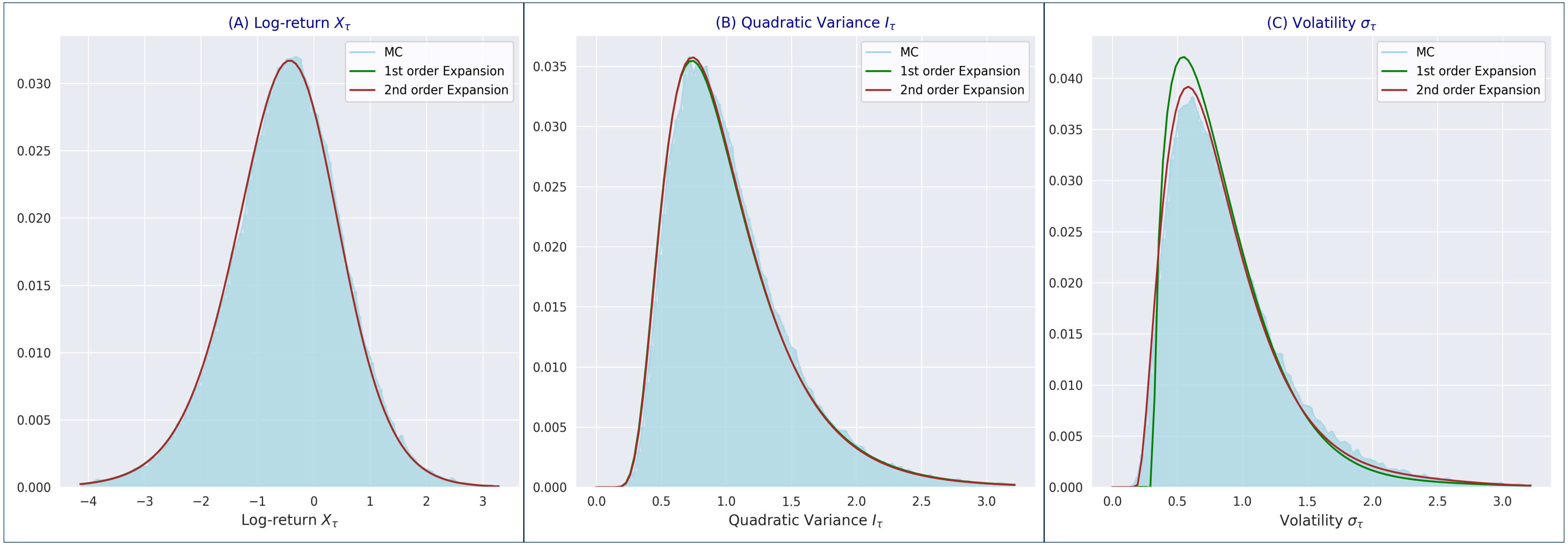
Corollary 3. *[Second-order affine approximation for the MGF (13)] is obtained using the leading term* $E^{[4]}$*:*

$$G(\tau, \Phi, \Psi, \Theta; p) = E^{[4]}(\tau, \Phi, \Psi, \Theta; p) \tag{17}$$

The approximation error is estimated using remainder $R^{[4]}$

Theorem 9. $E^{[4]}$ *is consistent with first and second moments of* (X, Y, I).

- Illustration of PDFs computed by inversion of the first $E^{[2]}$ and second order $E^{[4]}$ compared to Monte-Carlo histogram (blue) for $\tau = 1$



Extension to rough dynamics (Sepp-Rakhmonov-Motuzenko (2024) in progress)

- We consider dynamics driven by *log-normal Volterra SV*

$$
\begin{aligned}
dS_t &= r(t)S_t dt + \eta(t)\sigma_t S_t dW_t^{(0)}, \\
\sigma_t &= \sigma_0 + \int_0^t K(t-s)\left((\kappa_1 + \kappa_2\sigma_s)(\theta - \sigma_s)ds + \beta\sigma_s\, dW_s^{(0)} + \varepsilon\sigma_s\, dW_s^{(1)}\right)
\end{aligned}
\tag{18}
$$

where $W^{(0)}$, $W^{(1)}$ are uncorrelated Brownian motions
$\eta(t)$ is volatility backbone to fit to the term structure of variance swaps
Kernel $K : \mathbb{R}^+ \to \mathbb{R}^+$ is a fractional kernel

$$
K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1},\ \alpha \in \left(\frac{1}{2}, 1\right] \tag{19}
$$

- Boundary points $\{0, +\infty\}$ are unattainable for process σ_t

- σ_t has unique strong solution

- Dynamics (18) are closed (have the same functional form) under numeraire changes and under the inverse measure

Extension to rough dynamics II

- Analytic/numerical solution for expected quadratic variance for $\kappa_2 = 0$ / $\kappa_2 > 0$ for fitting volatility backbone $\eta(t)$

- Affine first/second order expansion can be applied to derive an exponential approximation to MGF of rough log-normal SV model

- However, the expansion results in a multi-variate system of integral equations which is numerically tedious

- Implement Monte-Carlo based valuation

- Develop Deep Learning (DL) for model calibration

Rough dynamics and volatility auto-correlation

• Auto-correlation of the volatility measures the "memory" or the longevity of periods with high volatility

• Auto-correlation function (ACF) of the volatility path σ_t observed at regular sampling times $\{t_n\}$

$$ACF(h) = Corr(\sigma_{t+h}, \sigma_t), \tag{20}$$

where h is the lag

• One-factor Markovian SV models produce exponentially decaying ACF:

$$ACF(h) = ce^{-qh}, \ |h| \to 0 \tag{21}$$

where $c > 0$ and $q > 0$ are constants

• Bennedsen et al (2022) suggest that rough SV models produce the following ACF:

$$ACF(h) = 1 - c\,|h|^{2\alpha'+1}, \ |h| \to 0, \tag{22}$$

where $c > 0$ is a constant and $\alpha' = \alpha - 1$, $\alpha' \in (-1/2, 1/2)$, is called the roughness index of σ_t

• Negative values of α' imply auto-correlation rougher than that of a Brownian motion

Estimation of rough auto-correlation

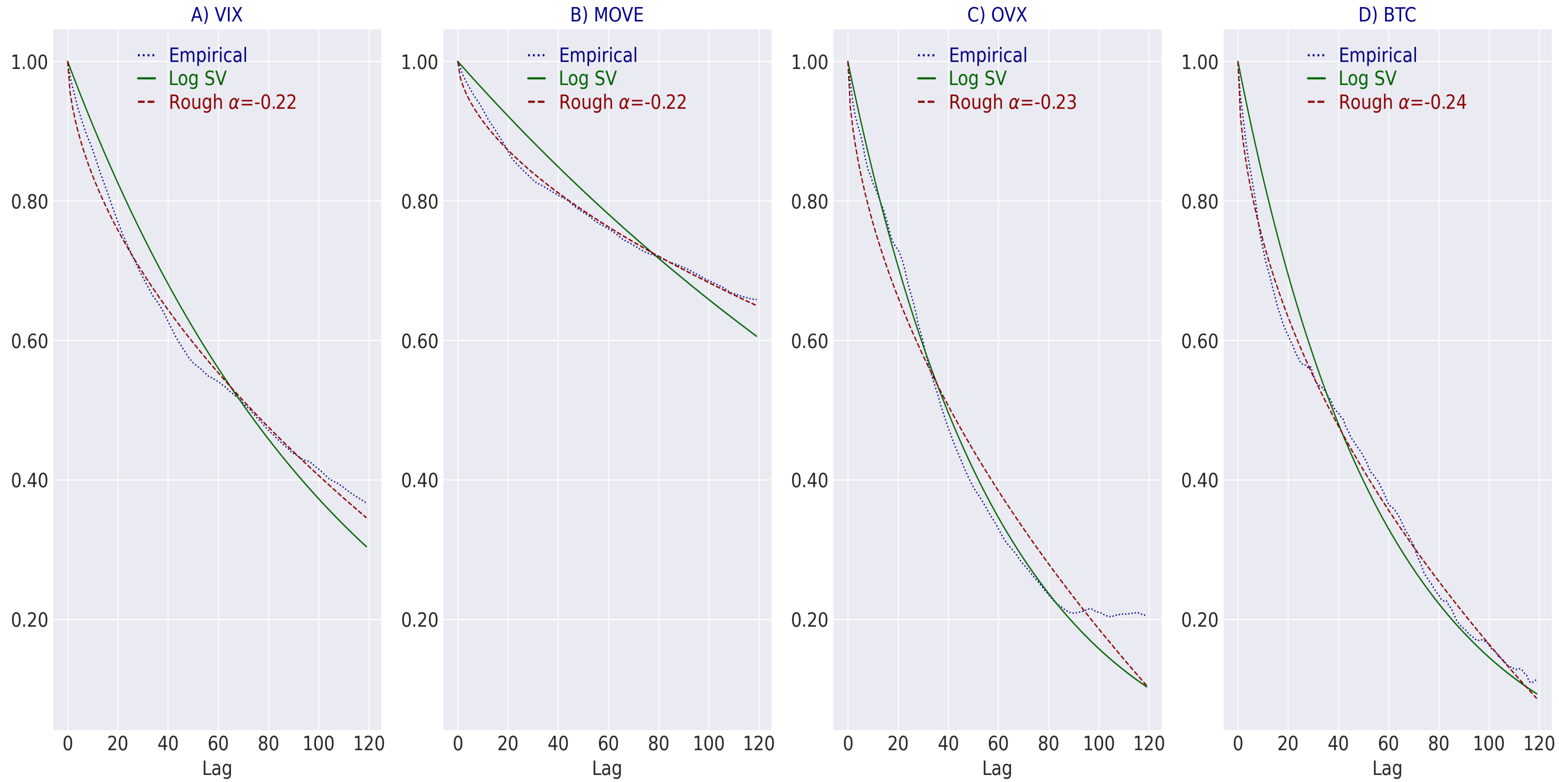
- We estimate the values of c and α' by minimizing squired differences empirical ACF and ACF given in Eq (22)

- To fit ACF implied by log-normal SV model, we fit model ACF computed by Eq (20) with MC simulations and fit stedy-state empirical distribution:

$$Model\ ACF(h) = \mathbb{E}^{\mathbb{P}}\left[Corr(\sigma_{t+h}, \sigma_t)\right] \tag{23}$$

for a vector of lags $h = (0, 1, ...)$

- Figure showing ACF of implied vols: A) VIX for the S&P 500 index, B) MOVE for 10y UST rate, C) OVX for oil ETF, D) BTC ATM vols



Remarks on model calibration to time series data of option prices

- One-factor Log-normal SV model is sufficient enough for reproducing the empirical features of ACF for daily data

- When fitting model parameters to options, it is hard to estimate the volatility mean-reversion parameters consistently because a higher value of mean reversion requires higher values of model correlation and convexity and vice versa

- Instead, we estimate the mean-reversion to the empirical ACF and keep it fixed when fitting model to options data

- There is a small correction in mean-reversion parameters between $\mathbb{P}$ and $\mathbb{Q}$ measures due to risk-premia, which we ignore for time being

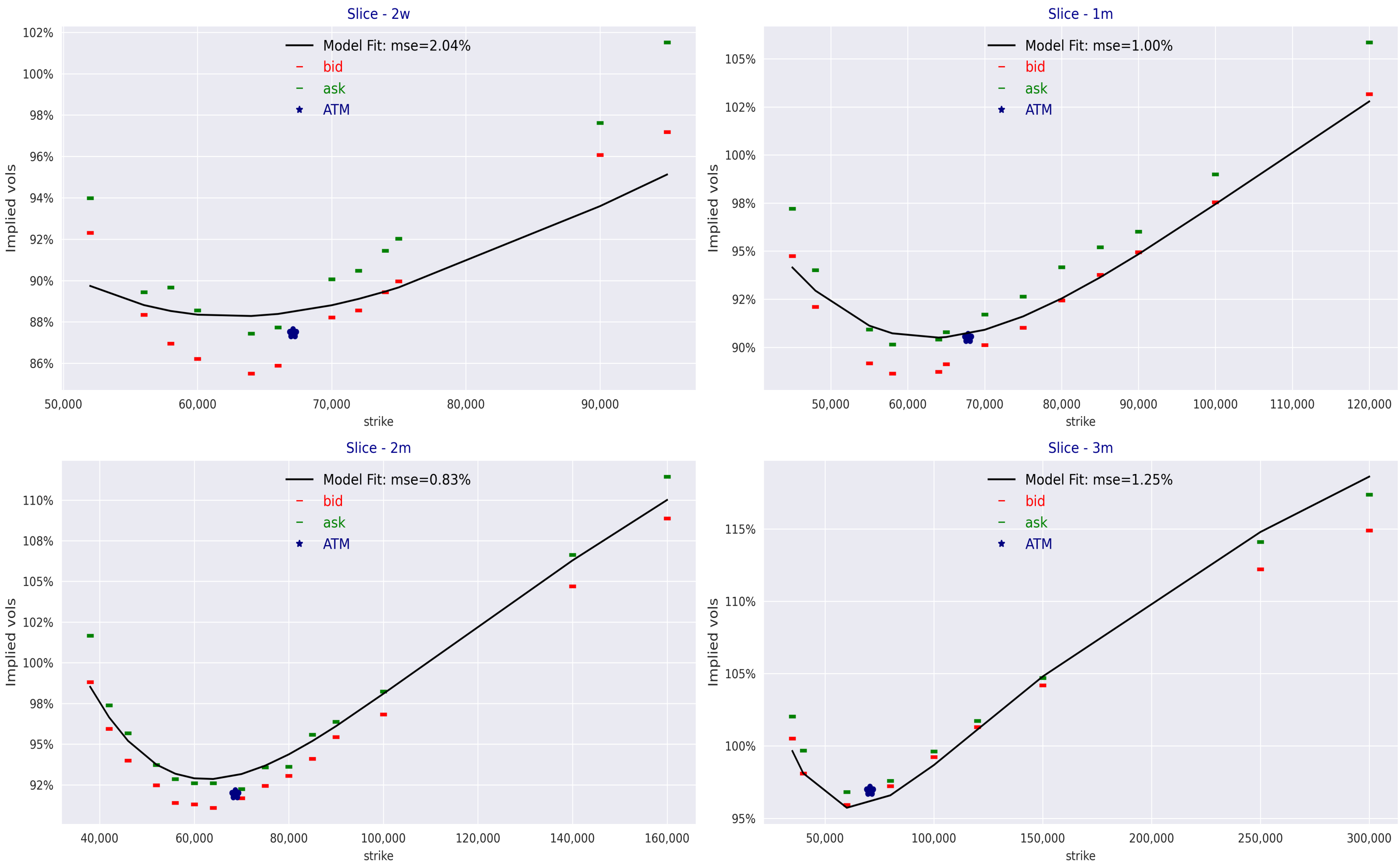
- Fitted parameters of steady-state distribution of volatility under Log-normal SV model

	VIX	MOVE	OVX	BTC
θ	0.20	0.91	0.39	0.71
κ_1	1.29	0.10	2.78	2.21
κ_2	1.93	0.41	2.24	2.18
ε	0.72	0.36	0.83	0.92

Calibration of model parameters to Bitcoin implied volatilities

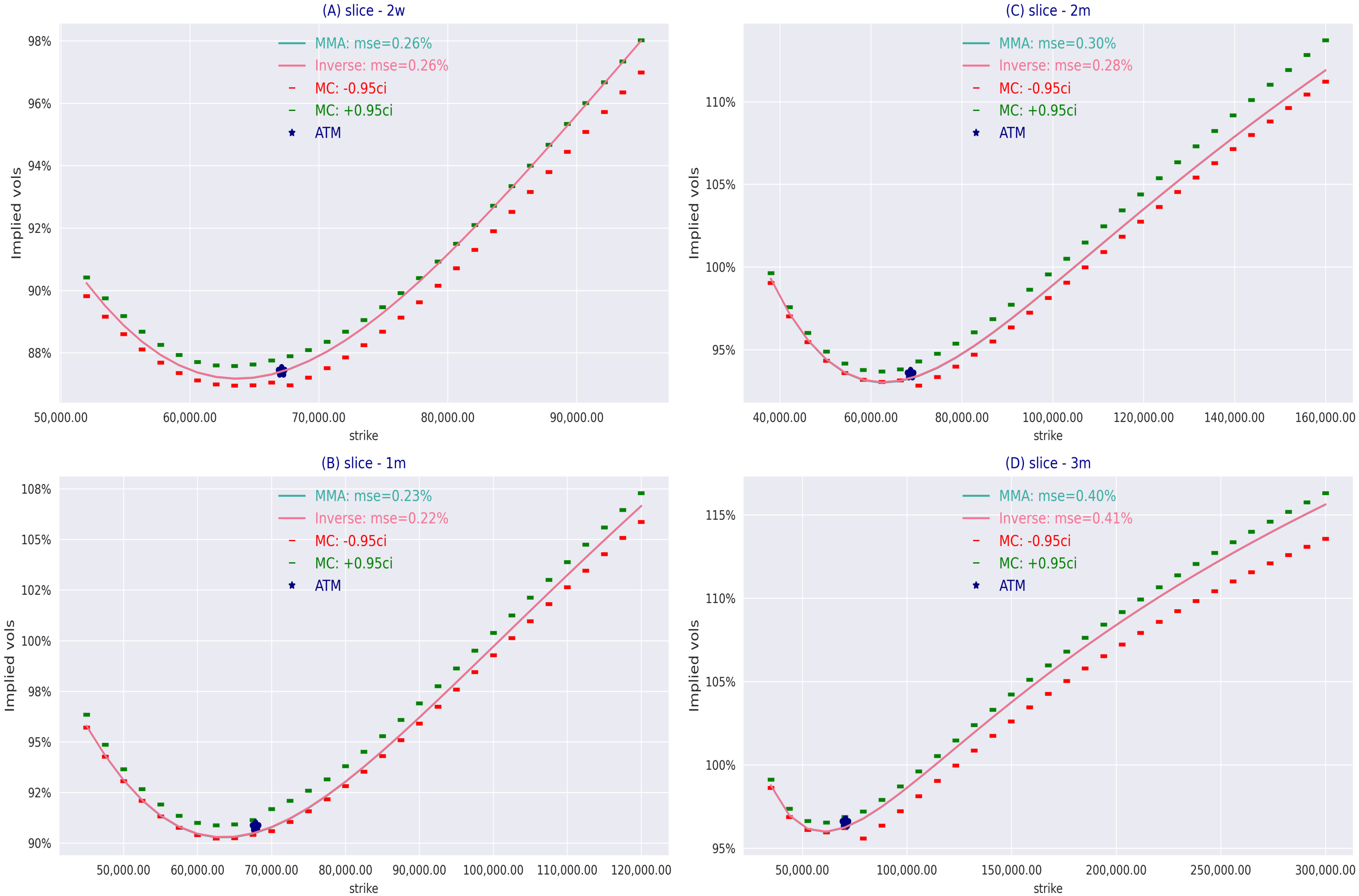
• Analytic solution for MGF provides Fourier-based formulas (Lewis (2000) and Lipton (2001) for vanilla options of extended to inverse options

• Model provides very good fit to market data within bid/ask
$\widehat{\sigma_0} = 0.86,\ \widehat{\theta} = 1.04,\ \widehat{\kappa_1} = 2.21,\ \widehat{\kappa_2} = 2.18,\ \widehat{\beta} = 0.13,\ \widehat{\varepsilon} = 1.63$



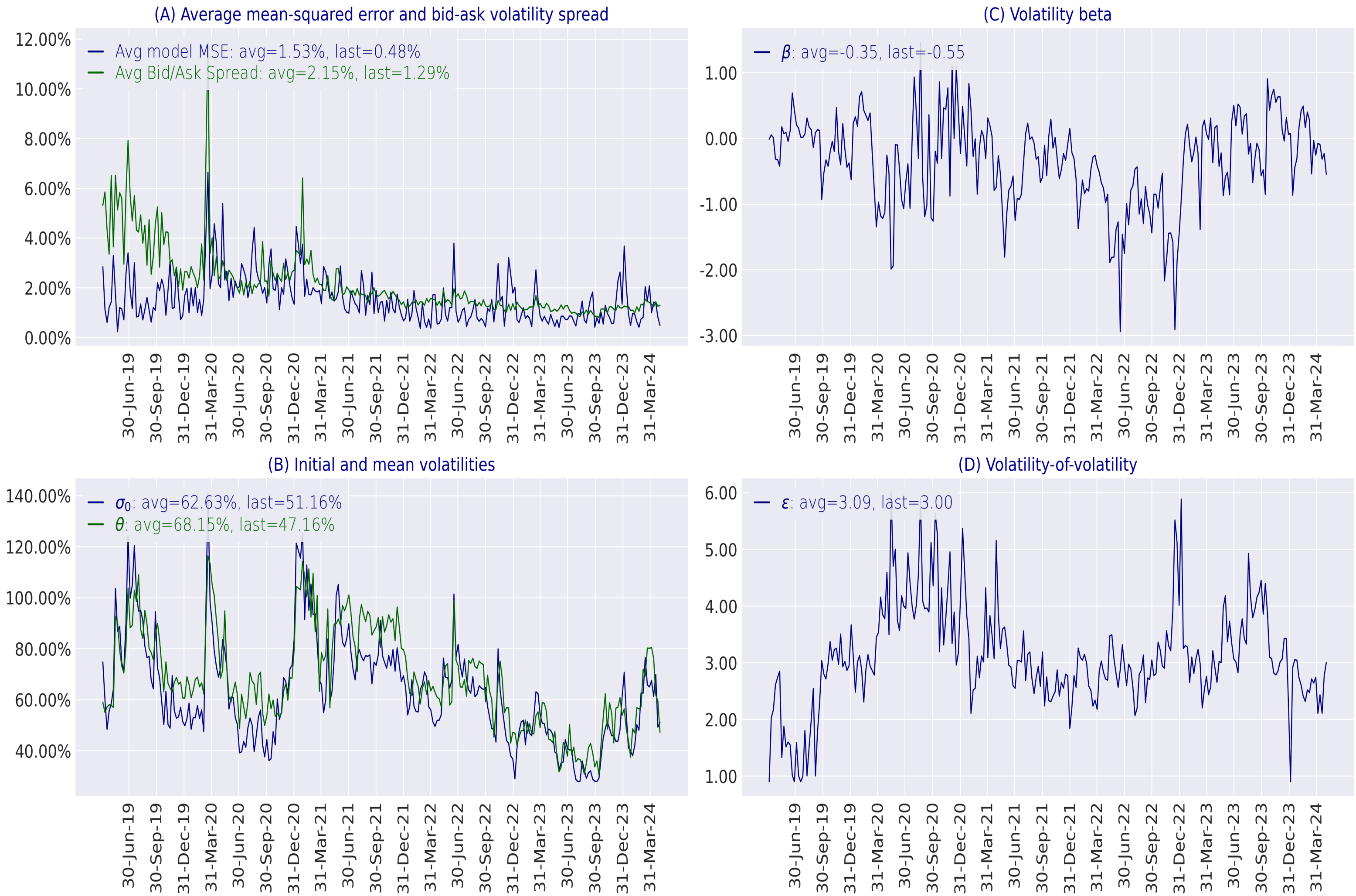
Analytic solution matches Monte-Carlo simulations

- Compute model implied volatilities under MMA measure (MMA) and inverse measure (Inverse) for maturity slices used in model calibration
- Dashed lines $MC - 0.95ci$ and $MC + 0.95ci$ are MC 95% CI
- MSE is the mean squared error between model and MC vols



Time series of model params fitted to short-dated BTC options

In 2022 and 2023, the model error (the average difference between market and model implied volatility) became less than 1% most of the times



Python library

Python implementation `https://github.com/ArturSepp/StochVolModels`

See `stochvolmodels/examples/run_lognormal_sv_pricer`

```python
# 1. create instance of pricer
logsv_pricer = LogSVPricer()

# 2. define model params
params = LogSvParams(sigma0=1.0, theta=1.0, kappa1=5.0, kappa2=5.0, beta=0.2, volvol=2.0)

# 3. compute model prices for option slices
model_prices, vols = logsv_pricer.price_slice(params=params,
                                              ttm=0.25,
                                              forward=1.0,
                                              strikes=np.array([0.8, 0.9, 1.0, 1.1]),
                                              optiontypes=np.array(['P', 'P', 'C', 'C']))
print([f"{p:0.4f}, implied vol={v: 0.2%}" for p, v in zip(model_prices, vols)])

# 4. calibrate model to test option chain data
btc_option_chain = get_btc_test_chain_data()
params0 = LogSvParams(sigma0=0.8, theta=1.0, kappa1=2.21, kappa2=2.18, beta=0.15, volvol=2.0)
btc_calibrated_params = logsv_pricer.calibrate_model_params_to_chain(option_chain=btc_option_chain,
                                                                      params0=params0,
                                                                      model_calibration_type=LogsvModelCalibrationType.PARAMS4,
                                                                      constraints_type=ConstraintsType.INVERSE_MARTINGALE)
print(btc_calibrated_params)

# 5. plot model implied vols
logsv_pricer.plot_model_ivols_vs_bid_ask(option_chain=btc_option_chain,
                                         params=btc_calibrated_params)
```

Conclusions

1. Introduce log-normal stochastic volatility model with quadratic drift to enforce martingality under money-market account and inverse measures

2. Develop closed-form affine expansion for accurate valuation of vanilla and inverse options

3. Discuss rough log-normal SV model and the impact of roughness on model auto-correlation function

4. Apply model for empirical fit to time series of option markets on Bitcoin with close fit to market data

Further reading

- Sepp A and Rakhmonov P (2023) Log-normal Stochastic Volatility Model with Quadratic Drift, International Journal of Theoretical and Applied Finance, 2023, 26(8) https://www.worldscientific.com/doi/reader/10.1142/S0219024924500031

- Lucic V and Sepp A (2024) Valuation and Hedging of Cryptocurrency Inverse Options, Quantitative Finance, 2024, 24(7), 851-869, https://ssrn.com/abstract=4606748

- Sepp A and Rakhmonov P (2023) What Is a Robust Stochastic Volatility Model, SSRN preprint https://ssrn.com/abstract=4647027

- Sepp A, Rakhmonov P, Motuzenko (2024) Rough Log-normal Stochastic Volatility Model, in preparation

Disclosure

These slides and discussion represent my personal views.

These views do not represent an official view of my current and last employers.

These views and discussion are not an investment advice in any possible form.

Volatility products and cryptocurrencies are associated with high risk.