Realized and implied index skews, jumps, and the failure of the minimum-variance hedging

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Plan

1) Empirical evidence for the log-normality of implied and realized volatilities of stock indices

2) Apply the beta stochastic volatility (SV) model for quantifying implied and realized index skews

3) Origin of the premium for risk-neutral skews and its impacts on profit-and-loss (P&L) of delta-hedging strategies

4) Optimal delta-hedging strategies to improve Sharpe ratios

5) Log-normal beta SV model
References
Technical details can be found in references

**Beta stochastic volatility model:**
http://ssrn.com/abstract=2150614


**Implied and realized skews, jumps, delta-hedging P&L:**
http://ssrn.com/abstract=2387845

http://ssrn.com/abstract=2522425

**Optimal delta-hedging strategies:**
http://ssrn.com/abstract=1865998
How to build a dynamic model for volatility?

Suppose we know nothing about stochastic volatility

We want to learn only by looking at empirical data

How do we start?
Empirical frequency of implied vol is log-normal

First, check whether stationary distribution of volatility is:
A) Normal or B) Log-normal

Compute the empirical frequency of one-month implied at-the-money (ATM) volatility proxied by the VIX index for last 20 years

Daily observations normalized to have zero mean and unit variance

Left figure: empirical frequency of the VIX - it is definitely not normal

Right figure: the frequency of the logarithm of the VIX - it does look like the normal density (especially for the right tail)!
Empirical frequency of realized vol is log-normal

Compute one-month realized volatility of daily returns on the S&P 500 index for each month over non-overlapping periods for last 60 years from 1954

Below is the empirical frequency of normalized historical volatility

**Left figure:** frequency of realized vol - it is definitely not normal

**Right figure:** frequency of the logarithm of realized vol - again it does look like the normal density (especially for the right tail)
Dynamic model for volatility evolution should not be based on price-volatility correlation

Now we look for a dynamic factor model for volatility (next slide)

We cannot apply model based on correlation between S&P500 returns and changes in volatility because using correlation we can only predict the direction of change, not the magnitude of change

For risk management of options, we need a factor model for volatility dynamics
Factor model for volatility uses regression model for changes in vol $V(t_n)$ predicted by returns in price $S(t_n)$

$$V(t_n) - V(t_{n-1}) = \beta \left[ \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})} \right] + V(t_{n-1})\epsilon_n$$  \hspace{1cm} (1)

iid normal residuals $\epsilon_n$ are scaled by vol $V(t_{n-1})$ due to log-normality

**Volatility beta** $\beta$ explains about 70% of variations in volatility!

**Left figure**: scatter plot of daily changes in the VIX vs returns on S&P 500 for past 14 years and estimated regression model

**Right**: time series of empirical residuals $\epsilon_n$ of regression model (1)

Residual volatility does not exhibit any systemic patterns

Regression model is stable across different estimation periods

**Volatility beta** $\beta$: expected change in ATM vol predicted by price return
For return of $-1\%$: expected change in vol $= -1.08 \times (-1\%) = 1.08\%$
More evidence on log-normal dynamics of vol: independence of regression parameters on level of ATM vol

Estimate empirically the elasticity $\alpha$ of volatility by:

1) computing volatility beta and residual vol-of-vol for each month using daily returns within this month

2) test if the logarithm of these variables depends on the log of the VIX in that month using regression model

Left figure: test $\hat{\beta}(V) = \beta V^\alpha$ by regression model: $\ln |\hat{\beta}(V)| = \alpha \ln V + c$

Right: test $\hat{\varepsilon}(V) = \varepsilon V^{1+\alpha}$ by regression model: $\ln |\hat{\varepsilon}(V)| = (1+\alpha) \ln V + c$

The estimated value of elasticity $\alpha$ is small and statistically insignificant. Indeed the realized volatility is close to log-normal.
Empirical estimation of volatility elasticity $\alpha$: volatility dynamics is log-normal
(maximum likelihood estimation - see my paper on log-normal volatility)

Figure: 95% confidence bounds for estimated value of elasticity $\alpha$ using realized (RV) and implied (IV) volatilities for 4 major stock indices

Estimation results confirm evidence for log-normality of volatility:
[i] In majority of cases (7 out of 12), bounds for $\hat{\alpha}$ contain zero
[ii] One outlier $\hat{\alpha} = -0.4$ (realized volatility of Nikkei index)
[iii] Remaining are symmetric: two with $\hat{\alpha} \approx 0.2$ and two with $\hat{\alpha} \approx -0.2$

To conclude - alternative SV models are safely rejected:
1) Heston and Stein-Stein SV models with $\alpha = -1$
2) $3/2$ SV model with $\alpha = 1$

Also, excellent econometric study by Christoffersen-Jacobs-Mimouni (2010), Review of Financial Studies: log-normal SV outperforms its alternatives
Beta stochastic volatility model (Karasinski-Sepp 2012): is obtained by summarizing our empirical findings for dynamics of index price \( S(t) \) and volatility \( V(t) \):

\[
dS(t) = V(t)S(t)dW^{(0)}(t)
\]
\[
dV(t) = \beta \frac{dS(t)}{S(t)} + \varepsilon V(t)dW^{(1)}(t) + \kappa (\theta - V(t))dt
\]

\( V(t) \) is either returns vol or short-term ATM implied vol
\( W^{(0)}(t) \) and \( W^{(1)}(t) \) are independent Brownian motions
\( \beta \) is volatility beta - sensitivity of volatility to changes in price
\( \varepsilon \) is residual vol-of-vol - standard deviation of residual changes in vol

Mean-reversion rate \( \kappa \) and mean \( \theta \) are added for stationarity of volatility

A closer inspection shows that these dynamics are similar to other log-normal based SV models widely used in industry:
A) in interest rates - SABR model
B) in equities - a version of log-normal based aka exp-OU SV models

We arrived to beta SV model \( \boxed{2} \) only by looking at empirical data for realized&implied vols and using factor model for vol dynamics
**Implied interpretation of volatility beta and residual vol-of-vol** from Black-Scholes-Merton (BSM) volatilities, $\sigma_{BSM}(z)$ as functions of log-strike $z = \ln(K/S)$, inferred from option prices.

Compute vol skew SKEW and convexity CONV for small maturities:

$$SKEW = [\sigma_{BSM}(5\%) - \sigma_{BSM}(-5\%)] / (2 \times 5\%)$$
$$CONV = [\sigma_{BSM}(5\%) + \sigma_{BSM}(-5\%) - 2\sigma_{BSM}(0)] / (5\%)^2$$

Volatility beta $\beta^{[I]}$ implied by skew:

$$\beta^{[I]} = 2 \times SKEW$$

Residual vol-of-vol $\varepsilon^{[I]}$ implied by convexity:

$$\varepsilon^{[I]} = \sqrt{3 \times \sigma_{BSM}(0) \times CONV + 2 \times (SKEW)^2}$$

As model parameters, volatility beta (**left figure**) and idiosyncratic vol-of-vol (**right figure**) have orthogonal impact on BSM implied vols.
Topic II: Implied and realized skew using beta SV model

Use time series from April 2007 to December 2013 for one-month ATM vols and the S&P500 index with estimation window of one month

**Figure 1):** Implied and realized one month volatilities
ATM volatility tends to trade at a small premium to realized

**Figure 2):** One-month average of implied and realized volatility beta
Implied volatility beta consistently over-estimates realized one

**Figure 3):** Average of implied and realized residual vol-of-vol
Implied residual vol-of-vol significantly over-estimates realized

Absolute (Abs) and relative (Rel) spreads between implieds&realizeds

<table>
<thead>
<tr>
<th>Spreads</th>
<th>Vol</th>
<th>Beta</th>
<th>VolVol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abs, Mean</td>
<td>0.51%</td>
<td>-0.27</td>
<td>0.78</td>
</tr>
<tr>
<td>Abs, Stdev</td>
<td>6.2%</td>
<td>0.21</td>
<td>0.16</td>
</tr>
<tr>
<td>Rel, Mean</td>
<td>7%</td>
<td>21%</td>
<td>57%</td>
</tr>
<tr>
<td>Rel, Stdev</td>
<td>24%</td>
<td>17%</td>
<td>11%</td>
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</table>
Explanation of the skew premium in a quantitative way

In a very interesting study, Bakshi-Kapadia-Madan (2003), Review of Financial Studies, find relationship between risk-neutral and physical skew using investor’s risk-aversion

Fat tails (not necessarily skewed) of returns distribution under physical measure $\mathbb{P}$ along with risk-aversion lead to increased negative skeweness under the risk neutral-measure $\mathbb{Q}$

Quantitatively:

$$SKEWENESS^Q = SKEWENESS^P - \gamma \times KURTOSIS^P \times VOLATILITY^P$$

$SKEWENESS^Q$ is risk-neutral skeweness of price returns
$SKEWENESS^P$ is physical skeweness of price returns
$KURTOSIS^P$ is kurtosis as measure of fat tails of physical distribution
$VOLATILITY^P$ is volatility of returns under physical distribution
$\gamma > 0$ is risk-aversion parameter of investors

To conclude: the risk-neutral premium arises because risk-averse investors assign higher value to insurance puts

Important: Volatility skew is proportional to skeweness of returns
Apply Merton Jump-Diffusion (JD) with normal jumps

Figure 1: Use last 14 years of daily returns on S&P 500 index to estimate skeweness and kurtosis of returns - see column "Empirical $P$"

Table 1: Use $\gamma = 22.0$ (estimated from time series of implied vols by inverting BKM formula) and apply BKM to obtain SKEWENESS $Q = -2$

Figure & Table 2: Fit Merton JD to first four moments of physical and risk-neutral distribution (jump frequency is set to one jump per month)

From calibration: JumpMean is $0$ under empirical $P$ and $-5\%$ under $Q$

<table>
<thead>
<tr>
<th></th>
<th>Empirical $P$</th>
<th>$Q$</th>
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<tbody>
<tr>
<td>Stdev</td>
<td>21%</td>
<td>21%</td>
</tr>
<tr>
<td>Skeweness</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>8</td>
<td>8</td>
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</table>

Merton JD params

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jump Mean</td>
<td>0%</td>
<td>-5%</td>
</tr>
<tr>
<td>Jump Volatility</td>
<td>4%</td>
<td>0%</td>
</tr>
<tr>
<td>Diffusion vol</td>
<td>17%</td>
<td>13%</td>
</tr>
<tr>
<td>Jump Frequency</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Figure 3: Value one month options - implied volatility from Merton JD under $Q$ is skewed, while implied volatility under $P$ is symmetric
To summarize our developments so far:

1) Log-normal beta SV model is consistent with empirical distribution for realized and implied vols

2) Beta SV model is applied to quantify realized and implied skews and the spread between them, which turns out to be significant

Any option position is mark-to-market so no point of arguing about market prices
However, hedging strategy is discretionary and can be the "edge"

By computing the delta-hedge: should we use implied or realized skews?

This question is analyzed in the third topic of my talk:

**Part I** - Quantitative analysis of impact of realized and implied skews on delta-hedging P&L

**Part II** - Monte-Carlo simulations for empirical analysis
Statistically significant spread between realized and implied skews $\beta^R - \beta^I$ leads to dependence on realized price returns and invalidates the minimum-variance hedge.

Minimum-variance delta $\Delta$ is applied to hedge against changes in price and price-induced changes in volatility.

Given hedging portfolio $\Pi$ for option $U$ on $S$

$$\Pi(t, S, V) = U(t, S, V) - \Delta \times S$$

$\Delta$ is computed by minimizing variance of $\Pi$ using SV beta dynamics (2) under risk-neutral measure $\mathbb{Q}$ (classic approach) with implied vol beta $\beta^I$

$$\Delta = U_S + \beta^I \times U_V / S$$

where $U_S$ and $U_V$ model delta and vega.

To see dependence on return $\delta S$ due to spread between implied vol beta $\beta^I$ and realized $\beta^R$: given $\delta S$ apply beta SV for change in vol $\delta V$ under physical measure $\mathbb{P}$:

$$\delta V = \beta^R \times \delta S + \varepsilon^R \times V \sqrt{\delta t}$$

By Taylor expansion of realized P&L:

$$\delta \Pi(t, S, V) = \left[ \beta^R - \beta^I \right] \times U_V \times \delta S + \varepsilon^R \times U_V \times V \sqrt{\delta t} + O(dt)$$

$\varepsilon^R$ is random non-hedgable part from residual vol-of-vol $O(dt)$ part includes quadratic terms $(\delta S)^2$, $(\delta V)^2$, $(\delta S)(\delta V)$.
Volatility skew-beta is important for computing correct option delta

Figure 1) Apply regression model for time series of ATM vols for maturities $T = \{1m, 3m, 6m, 12m, 24m\}$ (m=month) to estimate regression volatility beta $\beta_{\text{REGRES}}(T)$ using S&P500 returns:

$$\delta \sigma_{\text{ATM}}(T) = \beta_{\text{REGRES}}(T) \times \delta S$$

Volatility beta for SV dynamics is instantaneous beta for very small $T$

Regression vol beta decays in log-$T$ due to mean-reversion: long-dated ATM vols are less sensitive in absolute values to price-returns

Figure 2) Implied vol skew for maturity $T$ has similar decay in log-$T$

Figure 3) Volatility skew-beta is regression beta divided by skew

$$\text{Vol Skew-Beta}(T) \propto \frac{\beta_{\text{REGRES}}(T)}{\text{SKEW}(T)}$$
Technical supplement to compute model implied skew-beta (omitted during the talk)

Using backward pricers and PDE:
1) Compute the term structure of ATM volatility $\sigma_{ATM}(S_0; T)$ and skew $\text{SKEW}(S_0; T)$, with strike width $\alpha \%$, implied by model parameters
2) Bump the spot price down by $\alpha \%$, $S_1 = (1 - \alpha \%)S_0$, and apply corresponding bumping rule for model state variables
   For the beta SV:
   $$V_1 \rightarrow V_0 + \beta \alpha, \quad \theta \rightarrow \theta + \frac{\beta}{2\kappa} \alpha$$
   (3)
3) Compute new term structure of ATM vols $\sigma_{ATM}(S_1; T)$
4) Compute model implied skew-beta
   $$\text{Skew-Beta}(T) = -\frac{\sigma_{ATM}(S_1; T) - \sigma_{ATM}(S_0; T)}{\alpha \times \text{SKEW}(S_0; T)}$$
   (4)

Using Monte-Carlo pricers:
1) Specify number of paths and simulate set of independent Brownians
2) Compute paths starting from $\{S_0, V_0\}$
2A) Evaluate term structure of ATM volatility $\sigma_{ATM}(K = S_0; T)$ and skew using $\sigma(K = S_1; T)$, both using Brownians in 1)
3) Compute paths starting from $\{S_1, V_1\}$ with $S_1 = (1 - \alpha \%)S_0$ and $V(1)$ bumped as in Eq (3), using Brownians in 1)
4) Evaluate ATM vols $\sigma_{ATM}(K = S_1; T)$ and skew-beta by Eq (4)
Volatility and Skew contribution to P&L - important for volatility positions with daily mark-to-market!

Mark BSM implied vol \( \sigma_{BSM}(K) \) in \%-strike \( K \) relative to price \( S(0) \):

\[
\sigma_{BSM}(K; S) = \sigma_{ATM}(S) + \text{SKEW} \times Z(K; S)
\]

\( Z(K; S) \) is log-moneyness relative to current price \( S \):

\[
Z(K; S) = \ln \left( \frac{K \times S(0)}{S} \right)
\]

SKEW < 0 is inferred from spread between call and put implied vols.

In practice, this form is augmented with extras for convexity and tails.

Any SV model implies quadratic form for implied vols near ATM strikes (Lewis 2000, Bergomi-Guyon 2012) so my approach for vol P&L is generic.

**Volatility P&L** arises from change in spot price \( S \to S \{1 + \delta S\} \):

\[
\delta \sigma_{BSM}(K; S) \equiv \sigma_{BSM}(K; S \{1 + \delta S\}) - \sigma_{BSM}(K; S) = \delta \sigma_{ATM}(S) + \text{SKEW} \times \delta Z(K; S)
\]

**First contributor to P&L**: change in ATM vol \( \delta \sigma_{ATM}(S) \):

\[
\delta \sigma_{ATM}(S) = \sigma_{ATM}(S \{1 + \delta S\}) - \sigma_{ATM}(S)
\]

**Second contributor to P&L**: change in log-moneyness relative to skew:

\[
\delta Z(K; S) = -\ln(1 + \delta S) \approx -\delta S
\]
Example of volatility and skew P&L with regression beta (omitted during the talk)

\[ \sigma_{ATM}(S(0)) = 15\%, \ \delta S = -1.0\%, \ \text{SKEW} = -0.5, \ \beta_{\text{REGRESS}} = -1.0 \]

It is very important how we keep log-moneyness \( Z(K; S) \):

1) For strikes re-based to new ATM level (forward-based strikes): \( S \rightarrow S\{1 + \delta S\} \) and log-moneyness does not change \( \delta Z(K; S) = 0 \)

P&L arises from change in ATM vol predicted by price return computed using \( \beta_{\text{REGRESS}} \):

\[ \delta\sigma_{BSM}(K) = \beta_{\text{REGRESS}} \times \delta S = -1.0 \times -1\% = 1\% \]

2) For strikes fixed at old ATM level (vanilla strikes with fixed \( S(0) \))

Thus log-moneyness changes by \( \delta Z(K; S) \approx -\delta S = 1\% \)

P&L is change in ATM vol adjusted for change in money-ness:

\[ \delta\sigma_{BSM}(K) = \beta_{\text{REGRESS}} \times \delta S + \text{SKEW} \times \delta S = 1\% + (-0.5) \times (1\%) = 0.50\% \]
Changes in skew are not correlated to changes in price and ATM vols - important for correct predict of vol and skew P&L

*Empirical observations yet again confirm log-normality dynamics!* (Using S&P500 data from January 2007 to December 2013)

**Figure 1: weekly changes in 100% – 95% skew vs price returns** for maturity of one month (left) and one year (right)
Regression slope = 0.13 (1m) & 0.03 (1y); \( R^2 = 0\% \) (1m) & 1\% (1y)

**Figure 2: weekly changes in 100% – 95% skew vs changes in ATM vols** for maturity of one month (left) and one year (right)
Regression slope = −0.15 (1m) & −0.06 (1y); \( R^2 = 0\% \) (1m) & 2\% (1y)
Volatility skew-beta combines the skew and volatility P&L together

Given price return $\delta S$:

$$S \rightarrow S \{1 + \delta S\}$$

Volatility P&L is computed by:

1) For strikes re-based to new ATM level

Log-moneyness does not change, $\delta Z(K; S) = 0$

P&L follows change in ATM vol predicted by regression beta and vol skew-beta:

$$\delta \sigma_{BSM}(K) \equiv \delta \sigma_{ATM}(S) = \beta_{REGRESS} \times \delta S$$

$$= \text{SKEWBETA} \times \text{SKEW} \times \delta S$$

2) For strikes fixed at old ATM level

Log-moneyness changes by $\delta Z(K; S') \approx -\delta S$

P&L is change in ATM vol adjusted for skew P&L:

$$\delta \sigma_{BSM}(K) \equiv \delta \sigma_{ATM}(S) - \text{SKEW} \times \delta S$$

$$= [\text{SKEWBETA} - 1] \times \text{SKEW} \times \delta S$$

Positive change in ATM vol from negative return is reduced by skew
Volatility skew-beta under minimum-variance approach is applied to compute min-var delta $\Delta$ for hedging against changes in price and price-induced changes in implied vol

**A)** We adjust option delta for change in implied vol at fixed strikes

**B)** The adjustment is proportional to option vega at this strike:

$$\Delta(K,T) = \Delta_{BSM}(K,T) + [\text{SKEWBETA}(T) - 1] \times \text{SKEW}(T) \times \nu_{BSM}(K,T)/S$$

$\Delta_{BSM}(K,T)$ is BSM delta for strike $K$ and maturity $T$

$\nu_{BSM}(K,T)$ is BSM vega, both evaluated at volatility skew

I classify volatility regimes using vol skew-beta for delta-adjustments:

$$\Delta(K,T) = \begin{cases} 
\Delta_{BSM}(K,T) + \text{SKEW}(T) \times \nu_{BSM}(K,T)/S, & \text{Sticky local} \\
\Delta_{BSM}(K,T), & \text{Sticky strike} \\
\Delta_{BSM}(K,T) - \text{SKEW}(T) \times \nu_{BSM}(K,T)/S, & \text{Sticky delta} \\
\Delta_{BSM}(K,T) + \frac{1}{2}\text{SKEW}(T) \times \nu_{BSM}(K,T)/S, & \text{Empirical S&P500} 
\end{cases}$$

"Shadow" delta is obtained using ratio $O$ (may be different from $1/2$):

$$\Delta(K,T) = \Delta_{BSM}(K,T) + O \times \text{SKEW}(T) \times \nu_{BSM}(K,T)/S$$

which is traders’ ad-hoc adjustment of option delta
Volatility skew-beta and vol regimes (also see Bergomi 2009):

\[
\text{SkewBeta} = \begin{cases} 
2, & \text{Sticky local regime: minimum-variance delta in SV and LV} \\
1, & \text{Sticky strike regime: BSM delta evaluated at implied skew} \\
0, & \text{Sticky delta regime: model delta in space-homogeneous SV}
\end{cases}
\]

Empirical estimates for skew-beta and its lower and upper bounds are found by regression model (see my paper)

In beta SV model, with empirical estimate of vol beta and adding jumps/risk-aversion to match skew premium, we fit empirical vol skew-beta:

1) S&P 500: empirical skew-beta of about 1.5
2) STOXX 50: strong skew-beta close to 2
3) NIKKEI: weak skew-beta is about 0.5

As result: beta SV model with jumps can produce the correct delta!
Second part of topic III: Monte Carlo analysis of delta-hedging P&L

Now let’s have some fun and do some number crunching!

We are going to simulate the market dynamics and compare hedging performance under different specifications of delta.

In next few slides I briefly discuss the methodology. Details are provided for the interested for self-studying.

Details are important to understand how to improve the performance of delta-hedging strategies. Application to actual market data produces equivalent conclusions.

In my talk, I will only discuss final results and conclusions.
Apply beta SV for dynamics under physical measure \( \mathbb{P} \):

1) Index price \( S(t) \),
2) Volatility of returns \( V_{ret}(t) \):
3) Short-term implied volatility \( V_{imp}(t) \):

\[
\begin{align*}
\text{d}S(t) &= V_{ret}(t)S(t)dW^{(0)}(t) \\
\text{d}V_{ret}(t) &= \kappa^P \left( \theta^P - V_{ret}(t) \right) dt + \beta^P V_{ret}(t)dW^{(0)}(t) + \varepsilon^P V_{ret}(t)dW^{(1)}(t) \\
\text{d}V_{imp}(t) &= \kappa^I \left( \theta^I - V_{imp}(t) \right) dt + \beta^I V_{imp}(t)dW^{(0)}(t) + \varepsilon^I V_{ret}(t)dW^{(1)}(t)
\end{align*}
\]

4) At-the-money (ATM) implied vol \( V_{atm}(t) \) is obtained by computing model implied ATM vol for maturity \( T \) using model dynamics for \( V_{imp}(t) \)

Important: Model parameters are estimated from time series by maximum likelihood methods - as a rule, parameters for returns vol \( [P] \) and for implied vol \( [I] \) are different

Here, apply the same parameters for clarity

<table>
<thead>
<tr>
<th>Physical for</th>
<th>Returns ( dV_{ret}(t) ), [P]</th>
<th>Vol ( dV_{imp}(t) ), [I]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_{i}(0) )</td>
<td>16%</td>
<td>16.75%</td>
</tr>
<tr>
<td>( \theta^I )</td>
<td>16%</td>
<td>16.75%</td>
</tr>
<tr>
<td>( \kappa^I )</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>( \varepsilon^I )</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( \beta^I )</td>
<td>-1.0</td>
<td>-1.0</td>
</tr>
</tbody>
</table>
Volatility and skew premiums are produced using BSM implied volatility, $\sigma_{BSM}(K)$, as function of % strike $K$ relative to $S(0)$:

$$\sigma_{BSM}(K) = V_{atm}(t) + \text{SKEW} \times \ln \left( \frac{K \times S(0)}{S(t)} \right)$$  \hspace{1cm} (5)

SKEW = $-0.5$ \textbf{is vol implied skew specified exogenously} by

<table>
<thead>
<tr>
<th>strike %</th>
<th>BSM vol $\sigma_{BSM}(K)$</th>
<th>$\sigma_{BSM}(K) - V_{rel}(0)$</th>
</tr>
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<tbody>
<tr>
<td>99%</td>
<td>17.25%</td>
<td>1.25%</td>
</tr>
<tr>
<td>100%</td>
<td>16.75%</td>
<td>0.75%</td>
</tr>
<tr>
<td>101%</td>
<td>16.25%</td>
<td>0.25%</td>
</tr>
<tr>
<td>Market Skew</td>
<td></td>
<td>-0.50</td>
</tr>
</tbody>
</table>

Important - option delta is computed using two models:
1) Beta SV model with market implied beta $\beta[I] = -1.1$
2) Beta SV model with empirical beta $\beta[I] = -1.0$ and jumps (risk-aversion) to price-in excessive skew $-1.1 - 1.0 = -0.1$ (discussed later)
Both SV models fit to market skew exactly!

[ii] Premium of implied vol to realized vol is:

$$16.75\% - 16\% = 0.75\%$$ (in line with empirical spread)

[iii] Premium of implied and empirical beta is:

$$\beta[I] - \beta[R] = -1.1 - (-1.0) = 0.1$$ (empirical is about $-0.2$)

As we saw using Madan-Merton fits, physical dynamics don’t need to have asymmetric jumps to produce skew premium - now, skew premium arises from excess kurtosis produced by empirical SV model for returns
Consistency with market skew does not guarantee fit to empirical dynamics

Both hedging models are consistent with market implied skew

However, we observe discrepancy:

**SV model with market implied beta, called Minimum variance hedge**
Implies vol skew-beta about $2.0$, which is inconsistent with empirical dynamics

**SV model with jumps and empirical beta, called Empirical hedge:**
Implies vol skew-beta about $1.6$, which is consistent with empirical dynamics

Important - no re-calibration along a MC path is applied:

Both hedging models are initially consistent with the market skew - as price $S(t)$ and vol $V_{imp}(t)$ change, both models remain very close to market skew

Log-normality assumption - independence of implied & realized skew from volatility - comes into play
Specification for trading in delta-hedged positions:

1) **Straddle** - short ATM put and call  
   **Figure 1:** P&L profile with Delta= 0 is function of realized return squared  

   **Important:** P&L/delta of straddle are not sensitive to realized/implied skew  
   - Benefits from small realized variance of price returns

2) **Risk-reversal** - short put with strike 99% and long call with strike 101% of forward  
   **Figure 2:** P&L profile with Delta= -0.8 is function of realized return  

   **Important:** P&L/delta of risk-reversal are very sensitive to realized/implied skew  
   - Benefits from small realized covariance of changes in price and ATM vol
Specification for notionals of delta-hedged positions

Notionals are normalized by $\text{CashGamma} = \frac{1}{2} \times (S^2) \times \text{OptionGamma}$

Notionals for straddle:

\[
\text{PutNotional}(t_n) = \text{CallNotional}(t_n) = -\frac{0.5}{\text{ATM CashGamma}(t_n)}
\]

Notionals for risk-reversal:

\[
\begin{align*}
\text{PutNotional}(t_n) &= -\frac{0.5 \times (V_{atm}(t_n))^2 T}{2\% \times \{\text{Put Vega}(t_n)\}} \\
\text{CallNotional}(t_n) &= +\frac{0.5 \times (V_{atm}(t_n))^2 T}{2\% \times \{\text{Call Vega}(t_n)\}}
\end{align*}
\]

where 2% comes from strike width $2\% = 101\% - 99\%$

Important: for Straddle, cash-gamma is 1.0
For Risk-reversal, the vanna (vega of delta) is 1.0
Monte-Carlo analysis: P&L accrual

Daily re-balancing at times $t_n, n = 1, \ldots, N$

At the end of each day, we roll into new position so straddle is at-the-money and risk-reversal has the same strike width

Realized P&L is P&L on hedges minus P&L on options position:

$$\text{P&L} = \sum_{n=1}^{N} \left\{ \Delta(t_{n-1}) [S(t_n) - S(t_{n-1})] \ight.$$ \[
- [\Pi(T - dt, S(t_n), V_{atm}(t_n)) - \Pi(T, S(t_{n-1}), V_{atm}(t_{n-1}))]\}

$\Pi(T, S(t_n), V_{atm}(t_n))$ is options position computed using BSM formula and implied volatility skew (5) with $V_{atm}(t_n), T = 1/12, dt = 1/252$

Transaction costs are 2bp ($k = 0.0002$) per delta-rebalancing:

$$\text{TC} = k |\Delta(t_0)| S(t_0) + k \sum_{n=1}^{N} |\Delta(t_n) - \Delta(t_{n-1})| S(t_n)$$

where $\Delta(t_n)$ is combined delta for newly rolled position

Important: P&L across different days and paths is maturity-time and strike-space homogeneous - robust for statistical inference!
Monte-Carlo analysis - final notes
Trade notional is 100,000,000$
Realized P&L and explanatory variables are reported in thousands of $

Option maturity: one month
Daily re-hedging with total for each path: \( N = 21 \)
P&L is annualized by multiplying by 12

Draw 2,000 paths and compute realized P&L and price return, variance, volatility beta for changes in price and ATM vol, etc

**Price and volatility paths are the same for straddle and risk-reversal and different hedging strategies**

**A)** Analyze realized delta-hedging P&L (Profit and Loss) by

[i] Realized P&L and its volatility, transaction costs

[ii] Sharpe ratios

**B)** P&L Explain using regression model with explanatory variables
What factors (realized variance, covariance, etc) contribute to P&L
1. Analysis of realized P&L for straddle

Figure left - realized P&L with no accounting for transaction costs
Right - realized P&L with transaction costs

Approximately, straddle P&L is spread between implied & realized vols:

\[ P&L = \Gamma \times \left\{ (V_{atm})^2 - (V_{ret})^2 \right\} \]
\[ = 100,000 \times \left\{ (16.75\%)^2 - (16.00\%)^2 \right\} = 246 \]

where \( \Gamma \) is cash-gamma notional in thousands $

Realized P&L little depends on the delta hedging strategy

*Important is that asset drift is zero, otherwise P&L-s for different hedging strategies have directional exposure to realized asset drift.*
2. Analysis of realized P&L for risk-reversal

Figure: left - realized P&L with no accounting for transaction costs
Right - realized P&L with transaction costs

Approximately, risk-reversal P&L is spread between implied and realized co-variance of price and vol returns:

\[
P&L = \mathcal{V} \times \left\{ -\text{SKEW} \times \left[ (V_{atm})^2 + (V_{ret})^2 \right] + \beta^R \times (V_{ret})^2 \right\} \\
= 100,211 \times \left\{ 0.5 \times \left[ (16.75\%)^2 + (16.00\%)^2 \right] - 0.88 \times (16.00\%)^2 \right\} = 431
\]

where \( \mathcal{V} \) is vanna notional in thousands $.

Again, realized P&L little depends on the delta hedging strategy when asset drift is zero.
3. Analysis of transaction costs

Transaction costs are $2bp$ per traded delta notional or $1$ per $5,000$. 

**Left figure: realized transaction costs**
1) Risk-reversal has higher transaction costs due to larger delta notional
2) Minimum variance hedge and empirical hedge imply about equal transaction costs for straddle
3) **Minimum variance hedge implies higher transaction costs for risk-reversal** because of over-hedging the put side

**Right figure: volatility of transaction costs**
Volatility is about uniform and very small compared to mean costs

![Graph showing realized and volatility of transaction costs]
4. Volatility of Realized P&L

Left figure: P&L volatility without accounting for transaction costs
Empirical hedge implies lower P&L volatility for:
[i] Risk-reversal (about 20%)
[ii] Straddle (about 2 – 3%)

Because Minimum Variance delta over-hedges for put side and make delta more volatile

Right figure: volatility of realized P&L accounting for costs
1) Transaction costs increase P&L slightly by about 1 – 2%
2) Contrast with reduction of realized P&L by about 50%
5. Sharpe ratios of realized P&L-s

Left figure: Sharpe ratios for delta-hedging P&L without accounting for transaction costs
Right figure: Sharpe ratios for P&L accounting for costs

1) For straddle, both Minimum Variance and Empirical hedges imply about the Sharpe ratio

2) For risk-reversal, Minimum Var hedge implies smaller Sharpe than Empirical hedge (by about 20%) because of higher P&L volatility and transaction costs
P&L Attribution to risk factors is applied to understand what factors contribute to P&L by using regression

\[
P&L = \alpha + s_1 X_1 + s_2 X_2 + s_3 X_3 + s_4 X_4 + s_5 X_5 + s_6 X_6 \quad (6)
\]

\(\alpha\) ("Alpha") is theta related P&L - P&L we would realize if nothing would move

\(X_1\) ("Var") is returns variance: \(X_1 = \sum \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right)^2\)

\(X_2\) ("VolChange") is change in ATM vol: \(X_2 = \sum (V_{atm}(t_n) - V_{atm}(t_{n-1}))\)

\(X_3\) ("Covar") is covariance: \(X_3 = \sum \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right) (V_{atm}(t_n) - V_{atm}(t_{n-1}))\)

\(X_4\) ("VarVol") is variance of vol changes: \(X_4 = \sum (V_{atm}(t_n) - V_{atm}(t_{n-1}))^2\)

\(X_5\) ("Return\(^3\)") is cubic return: \(X_5 = \sum \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right)^3\)

\(X_6\) ("Return") is realized return: \(X_6 = \sum \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right)\)

Summation \(\sum\) runs from \(n = 1\) to \(n = N\), \(N = 21\)

\(R^2\) indicates how well the realized variables explain realized P&L (not accounting for transaction costs) - we should aim for \(R^2 = 90\%\)

Some explanatory variables are correlated so it is robust to test reduced regressions
P&L explain for straddle by realized variance of returns: Empirical hedge has stronger explanatory power

Is needed to confirm theoretical P&L explain by MC simulations

For P&L of straddle hedged at implied vol, first-order approximation:

\[ V_{atm}^2 - \sum_n \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right)^2 \]

**First** term is alpha or "carry" - approximate alpha is
\[ \alpha = \Gamma \times V_{atm}^2 = 100,000 \times 0.1675^2 = 2806 \]

**Second** term is short risk to realized variance - key variable for P&L

Theoretical slope should be \(-\Gamma = -100,000\)

**Figure:** explanatory power using only realized variance is weak because of impact of other variables and skew (for multiple variables, \(R^2 \approx 90\%\))
P&L explain for risk-reversal by realized vol beta:
Empirical hedge implies that realized vol beta is clear
driver behind P&L of risk-reversal with $R^2 = 50\%$

For P&L of risk-reversal hedged at implied vol skew, approximation:

$$-\text{SKEW} \times \left\{ V_{atm}^2 + \sum_n \left[ \frac{S(t_n)}{S(t_{n-1})} - 1 \right]^2 \right\} + \sum_n \left( \frac{S(t_n)}{S(t_{n-1})} - 1 \right) (V_{atm}(t_n) - V_{atm}(t_{n-1}))$$

In terms of returns vol $V_{ret}$ and implied vol beta $\beta^R$:

$$-\text{SKEW} \times \left\{ V_{atm}^2 + V_{ret}^2 \right\} + \beta^R \times V_{ret}^2$$

**First** term is "carry" or alpha

**Second** term is risk to realized beta between returns and vol - key variable

In our example: $\alpha = 0.5 \times V \times \left\{ (16.75\%)^2 + (16.00\%)^2 \right\} = 2,682$

Slope$= V \times (16.00\%)^2 = 2,560$

**Risk-Reversal P&L by Min-Var Hedge**
P&L = 2129*Beta + 2570
$R^2 = 37\%$

**Risk-Reversal P&L by Empirical Hedge**
P&L = 2050*Beta + 2490
$R^2 = 49\%$
Important: vol beta (for skew) is comparable to Black-Scholes-Merton (BSM) implied volatility (for one strike)

1) Volatility and vol beta are meaningful and intuitive model parameters which can be inferred from both implied and historical data

- **Implied vol** $\sigma[I]$ is inferred from option market price
- **Realized vol** $\sigma[R]$ is volatility of price returns

- **Implied vol beta** $\beta[I]$ is inferred from market skew ($\beta[I] \approx 2 \times \text{SKEW}$)
- **Realized vol beta** $\beta[R]$ is change in implied ATM volatility predicted by price returns: $\beta[R] = \langle dS(t)dV_{atm}(t) \rangle / (\sigma[R])^2$

2) Both serve as directs input for computation of hedges

3) Both allow for P&L explain of vanilla options in terms of implied and realized model parameters:

- **Implied/realized volatility** - P&L of delta-hedged straddle:
  \[
  \left( \sigma[I] \right)^2 - \left( \sigma[R] \right)^2
  \]

- **Implied/realized volatility beta** - P&L of short delta-hedged risk-reversal (more noisy because of contribution from $\sigma[R]$):
  \[
  -\beta[I] \times \left\{ \frac{1}{2} \left[ \left( \sigma[I] \right)^2 + \left[ \sigma[R] \right]^2 \right] \right\} + \beta[R] \times \left( \sigma[R] \right)^2
  \]
Conclusion: existing practical approaches for hedging improvement are not fully satisfactory - we need proper model for dynamic delta-hedging!

A) Hedge all vega exposure

B) Recalibration for computing delta-risks (most common):
   ⊗ Project change in implied volatility using empirical backbone (For example, by applying empirical volatility skew-beta)
   ⊗ Re-calibrate valuation model to bumped volatility surface
   ⊗ Re-valuate and compute delta by finite-differences

However runs into problems:
1) A) - vega-hedging is (very) expensive and unprofitable unless implied skew and vol-of-vol are sold at large premiums to future realizeds

2) B) - re-calibration works poorly for path-dependent and multi-asset products and it makes P&L explain very noisy
   Recall applying regression for P&L explain of straddle and risk-reversal

3) any mix of A) and B) becomes very tedious for CVA computations

Important: the choice between local vol (LV) or stoch vol (SV) is irrelevant when hedging using minimum variance hedge at implied vol skew - any combination of LV and SV produces almost the same deltas!
Beta SV model with jumps is fitted to empirical & implied dynamics for computing correct delta (Sepp 2014):

\[
\frac{dS(t)}{S(t)} = (\mu(t) - \lambda(e^\eta - 1)) dt + V(t) dW^{(0)}(t) + (e^\eta - 1) dN(t)
\]

\[
dV(t) = \kappa(\theta - V(t)) dt + \beta V(t) dW^{(0)}(t) + \epsilon V(t) dW^{(1)}(t) + \beta \eta dN(t)
\]

1) Consistent with empirical dynamics of implied ATM volatility by specifying empirical volatility beta \( \beta \)

2) Has jumps, as degree of risk-aversion, to make model fit to both empirical dynamics and risk-neutral skew premium

Only one parameter with simple calibration! - explained in a bit

Jumps/risk-aversion under risk-neutral measure \( \mathbb{Q} \) produced by:

Poisson process \( N(t) \) with intensity \( \lambda \):

negative & positive jumps in returns & vols with constant size \( \eta < 0 & \beta \eta > 0 \)

3) Easy-to-implement (with no extra parameters) extension to multi-asset dynamics using common jumps - produces basket correlation skew

4) Beta SVJ model is robust to produce optimal hedges for path-dependent and multi-asset trades and CVA
Third to last topic: closed-form solution for log-normal Beta SV

Mean-reverting log-normal SV models are not analytically tractable

I derive a very accurate exp-affine approximation for moment generating function (details in my paper)

Idea comes from information theory: apply Kullback-Leibler relative entropy for unknown PDF $p(x)$ and test PDF $q(x)$ with moment constraints: 

$$
\int x^k p(x) dx = \int x^k q(x) dx, k = 1, 2, \ldots
$$

Now let’s think in terms of moment function:

[i] MGF for Beta SV model with normal driver for SV (as in Stein-Stein SV model) has exact solution, which has exp-affine form

[ii] Correction for log-normal SV has an exp-affine form
Proof that closed-form MFG for log-normal model produces theoretically consistent probability density

1) Derive solutions for expected values, variances, and covariances of the log-price and quadratic variance (QV) by solving PDE directly

2) Prove that moments derived using approximate MGF equal to theoretical moments derived in 1)

Using closed-form MFG for log-normal model, we apply standard valuation methods for affine SV models based on Lipton-Lewis formula

Implementation of closed-form moment function (MGF), MC, and PDE pricers produce values of vanilla options on equity and quadratic variance that are equal within numerical accuracy of these methods
Second to last topic - optimal hedging under discrete trading and transaction costs

As we saw in simulation of P&L, we need **quantitative framework that incorporates discrete hedging and optimizes trade-off between:**
the reward - higher P&L and lower transaction costs
the risk - higher P&L volatility
Illustration of trading in implied & realized vol with straddle: unique optimal hedging frequency can be found!

Figure 1) Forecast expected upside: the spread between implied and realized vol for given maturity $T$
This is independent of valuation & hedging model and hedging frequency

Figure 2) Forecast P&L volatility and transaction costs
These depend on valuation & hedging model and hedging frequency

Part of P&L volatility is not hedgeable due to vol-of-vol and jumps - Not optimal to hedge too frequently

Figure 3) Obtain Sharpe ratio as ratio of forecast P&L after costs and P&L volatility
Solution for optimal Sharpe ratio with dynamics under physical measure driven by Diffusion and SV with jumps

\[
\text{Sharpe}(N) = \frac{\text{Expected P&L} - \text{TransactionCosts}(N)}{\text{P&L Volatility}(N)}
\]

\(N\) is hedging frequency - for details see my paper on optimal delta-hedging

Using this solution we can analyze:

**Figure 1)** What maturity is optimal to trade given the forecast spread between implieds and realizeds
(longer maturities have higher spreads but their P&L is more volatile because of higher risk to ATM vol changes)

**Figure 2)** What is optimal hedging frequency for each maturity
Translate into approximations of optimal bands for price and delta triggers

Naturally, results are sensitive to assumed price dynamics
Under SV with jumps: lower Sharp ratio and less frequent hedging
Last topic: why the beta stochastic vol model with jumps is better than its alternatives (for stock indices)

The most important feature for dynamic hedging model:
1) Ability to produce different volatility regimes as observed in the market and to imply empirically consistent delta

Recall definition of volatility skew-beta: change in term structure of ATM volatility, $\sigma_{ATM}(T')$, predicted by price return times SKEW($T'$)

We saw that vol skew-beta is very important to account for correct P&L arising from change in BSM implied vols

Skew-consistent SV and LV models imply skew-beta of 2

**Empirical vol skew-beta:** S&P 500 $\approx$ 1.5; STOXX50 $\approx$ 1.8; Nikkei $\approx$ 0.5
Why the beta SV with jumps is better than its alternatives

Extra arguments to look at apart from implied volatility skew-beta

2) Fit to empirical distribution of implied and realized volatilities

3) Interpretation of model parameters in terms of impact on model implied BSM vols

4) P&L explain for delta-hedging strategies of vanilla options in terms of implied and realized model parameters

5) Stability of model parameters

Calibration to vanilla options is not a problem in practical applications - it is easy to achieve by introducing a (small) local vol part

I. Non-parametric local volatility model - textbook implementation of Dupire local volatility using discrete set of option prices and interpolation
II. Industry-standard alternative (in equity derivatives)

Implied volatility@strike-into-density@price approach (my terminology)

Conceptually:

\[ \sigma_{impl}(K; T) \rightarrow P_{impl}(S(T) = K) \]  

(7)

where \rightarrow is Dupire LV formula in terms of implied vols at strike \( K \) &mat \( T \)

Figure 1A) Given parametric form for implied vols \( \sigma_{impl}(K; T) \)

Figure 1B) Given backbone function \( f_{backbone}(\delta S; K, T) \) to map price changes \( \delta S \) into changes in vols \( \delta \sigma_{impl}(K; T) \) according to specified regime

Figure 2) \rightarrow in Eq(7) serves as interpolator from implied vols in strike space to implied densities in price space

Figure 3) LV model projects densities to option prices in ”model-independent” way using MC or PDE methods
1) Hedging performance for local vol approach are primarily driven by parametric form for implied vols $\sigma_{impl}(K; T)$ and empirical backbone function.

2) No consistency with empirical distribution of implied and realized vol.

3) & 4) Model interpretation and P&L explain are possible only in terms of parameters of functional form for implied volatility.

Key drawback of implied volatility-into-density approach:

[i] For computation of delta it requires a re-calibration of local vol and re-valuation for any change in market data.

[ii] Lacks vol-of-vol so it is inconsistent for hedging of path-dependent options sensitive to forward vols and skews.
Alternatives for local vol or $\sigma_{impl}(K; T) \rightarrow P_{impl}(S(T) = K)$ approach do not produce improvements

Instead of LV to map implied vol into price density, it is also customary to use SV or LSV models as interpolators with extra degree of freedom

Hereby model choice is typically motivated by availability of a "closed-form" solution, not empirical consistency!

Figure: SV and LSV models are not applied for hedging as dynamic models since their model delta is wrong - with and without minimum variance hedge - but through re-calibration to empirical backbone

To conclude I use a quote from Richard P. Feynman:
*It doesn’t matter how beautiful your theory is, it doesn’t matter how smart you are. If it doesn’t agree with experiment, it’s wrong!*
III. Arguments in favor of Beta SVJ model:
1) the model has ability to fit empirical vol skew-beta and produce correct option delta without re-calibration

Figure: delta from SVJ model fits empirical backbone

2) Consistent with the empirical distributions of implied and realized volatilities, which are very close to log-normal

3) It has clear intuition behind the key model parameters:
Volatility beta is sensitivity to changes in short-term ATM vol
Residual vol-of-vol is volatility of idiosyncratic changes in ATM vol

4) P&L explain is possible in terms of implied and realized quantities of key model parameter - vol beta

5) Stability and calibration - next slide
Calibration of beta SV model is based on econometric and implied approaches without large-scale non-linear and non-intuitive calibrations

1) Parameters of SV part are estimated from time series

2) Jump/risk-aversion params are fitted to empirical vol skew-beta

Params in 1) & 2) are updated only following changes in volatility regime

3) Small mis-calibrations of the SV part and jumps are corrected using local vol (LV) part

Contribution to skew from LV part is kept small (no more than 10-15%)

Local vol part is re-calibrated on the fly to reproduce small variations in some parts of implied vol surface, which are caused by temporary supply-demand factors specific to that part

It is also robust to compute bucketed vega risk in this way

In practical terms:
1) Local volatility part accounts for the noise from idiosyncratic changes in implied volatility surface

2) Stochastic volatility and jumps serve as time- and space-homogeneous factors for the shape of the implied volatility surface
More details on calibration of beta SV model (technical part omitted during the talk)

1) Parameters of SV part are calibrated using maximum likelihood methods from time series of 1m implied ATM volatility (or the VIX)
   [i] mean-reversion $\kappa$ is estimated over longer-period, at least 5 years, - better to keep it constant at 3.00
   [ii] vol beta $\beta$ and residual vol-vol $\varepsilon$ are estimated over shorter periods, 1y, - typically $\beta \approx -1.00$ and $\varepsilon \in [0.60, 1.00]$

2) Negative jump in return $\eta$ is fitted using Merton jump model to put options with maturity of 6 months and [80% – 100%] OTM strikes - typically $\eta = -30\%$

3) Given 1) and 2): 3A) jump intensity $\lambda$ is calibrated to fit the empirical sensitivity of implied volatility changes to price changes, aka volatility skew-beta, - typically $\lambda \in [0.03, 0.2]$
   Given all above: 3B) initial vol $V(0)$ and mean vol $\theta$ calibrated to fit the current term structure of ATM vols

Parameters in 1), 2) 3A) (relatively uniform for major stock indices) are updated infrequently

4) Local vol part is added to fit daily variations in implied vol surface
For risk-neutral pricing, distribution of jumps does not matter - jumps are only needed to fit skew premium.

Recall illustration of emergence of $Q$-skew using Bakshi-Kapadia-Madan formula and Merton jump model:

[i] Under $P$, jumps are symmetric with mean of 0% and volatility of 4%

[ii] The risk-neutral mean jump is −5% with zero volatility

Yet, jumps are needed to fit market prices & compute correct deltas.

Also jumps are important to fit market prices of options on realized and implied volatilities (VIX) - see my presentations at GD in 2011 & 2012

Practical explanation for excessive risk-neutral skew premium:

1) Risk-averse investors always ready to over-pay for insurance irrespectively of price changes.

2) As part of index correlation skew premium, when holders of stock portfolios buy index puts for (macro) protection.

To make things robust, I assume constant jumps with simple calibration.
How to explain the difference between implied and realized dynamics using preference theory

For retail option buyer - option value is derived from his preference/utility for specific payoffs in certain market scenarios

For institutional option seller - option value is derived from:

[i] Expected hedging costs
[ii] Smooth stream of fees and P&L
[iii] Premium for suffering losses in bad market conditions

As a result:

1) Option prices in the market are set by demand-supply equilibrium between sellers and buyers

2) Risk-aversion parameters is a degree of demand-supply imbalance

3) Implied and realized vols and, in particular, skews are different
To conclude: we can think of jumps as a measure of risk-aversion for pricing kernel!

Recently, interesting research is made and also presented at Global Derivatives on how to imply the "expected-implied" physical distribution from options market prices and specified risk-aversion.

Peter Carr: *Can we recover?*, Global Derivatives 2013

Computation of "empirical" delta and calibration of excessive skew are related concepts:
1) **Compute option delta under "expected-implied" physical distribution using empirical vol beta**
2) **Fit level of risk-aversion to excessive skew premium observed in market prices of index options**

These concepts and volatility skew-beta are related to the interplay between the implied and realized risk premiums:

**[i]** high implied / positive realized risk premiums - sticky strike vol regime

**[ii]** low implied / negative realized risk premiums - sticky local vol regime
Summary
1) Dynamics of implied and realized vols are log-normal
2) Implied vol beta significantly overestimates realized beta
3) Vol skew-beta is important for correct P&L - any dynamic hedging model should fit empirical skew-beta Risk-Aversion/Jumps parameter is added to fit empirical skew-beta SVJ fits empirical skew-beta ≈ 1.5, unlike Minimum Var delta ≈ 2.0
4) Beta SVJ model applied for delta-hedging risk-reversal is tool to produce P&L from spread between implied and realized skews Log-normal beta SVJ model: ☐ Is consistent with the empirical dynamics of ATM volatility ☐ Produces correct option deltas ☐ Can significantly improve Sharpe ratios for delta-hedging P&Ls
Disclaimer

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References