Achieving Consistent Modeling Of VIX and Equities Derivatives

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Plan of the presentation

1) Discuss model complexity and calibration

2) Emphasize intuitive and robust calibration of sophisticated volatility models avoiding non-linear calibrations

3) Present local stochastic volatility models with jumps to achieve joint calibration to VIX options and (short-term) S&P500 options

4) Present two factor stochastic volatility model to fit both the short-term and long-term S&P500 option skews

References

Some theoretical and practical details for my presentation can be found in:

1) Sepp, A. (2008) VIX Option Pricing in a Jump-Diffusion Model, Risk Magazine April, 84-89
`http://ssrn.com/abstract=1412339`

2) Sepp, A. (2008) Pricing Options on Realized Variance in the Heston Model with Jumps in Returns and Volatility, Journal of Computational Finance, Vol. 11, No. 4, pp. 33-70
`http://ssrn.com/abstract=1408005`

Motivation. Model complexity and calibration

The conventional approach emphasizes the importance of ”closed-form” solutions for a limited class of model chosen on the sole ground of their analytical tractability

The justification is that since the model calibration is implemented by a non-linear optimization, closed-form solution ”speed-up” the calibration

I prefer the opposite route:

1) Develop robust PDE methods for generic one and two factor stochastic volatility models

2) Obtain intuition about model parameters using approximation so that non-linear fits can be avoided

Model specification I

Changes in the implied volatility surface can be factored as follows:

$$\begin{aligned}\delta\sigma(T,K) &= \beta_1 \,\text{(change in short term ATM volatility)} \\ &+ \beta_2 \,\text{(change in term-structure of ATM volatility)} \\ &+ \beta_3 \,\text{(change in short term skew)} \\ &+ \beta_4 \,\text{(change in term-structure of skew)} \\ &+ \ldots\end{aligned} \tag{1}$$

Factors are reliable if:
1) They explain most of the variation
2) Can be estimated from historical and current data in intuitive ways
3) Can be explained in simple terms

Model specification II. Important factors for a volatility model
1) Short-term ATM volatility and term-structure of ATM volatility - time-dependent level of the model "ATM" volatility (the least of the problem)

2) Short-term skew - jumps in the underlying and volatility factor(s)

3) Term-structure of skew and volatility of volatility - mean-reversion and volatility of volatility of volatility factor(s)

The challenge is to re-produce these factors using stochastic volatility model with time-homogeneous parameters

In my presentation, I consider:

Part I - calibration of stochastic volatility with jumps to VIX skews (SV model for short-term skew, up to 6m)

Part II - calibration of two-factor stochastic volatility model (SV model for medium- and long-term skew, 3m-5y)

Part I. Joint calibration of SPX and VIX skews using jumps

I consider several volatility models to reproduce the volatility skew observed in equity options on the S&P500 index:
Local volatility model (LV)
Jump-diffusion model (JD)
Stochastic volatility model (SV)
Local stochastic volatility model (LSV) with jumps

For each model, I analyze its implied skew for options on the VIX

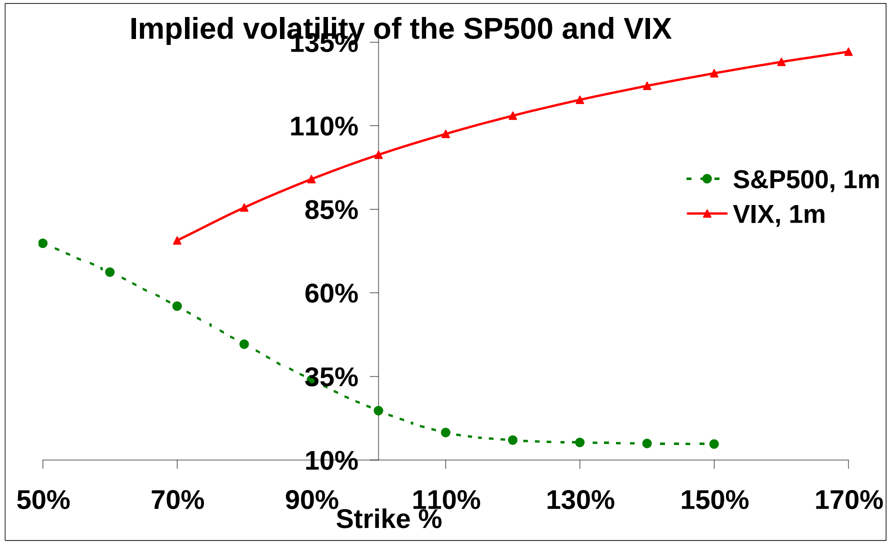
I show that LV, JD and SV without jumps are not consistent with the implied volatility skew observed in option on the VIX

I show that:
Only the SV model with appropriately chosen jumps can fit the implied VIX skew

Importantly, that only the LSV model with jumps can fit both Equity and VIX option skews

Motivation I

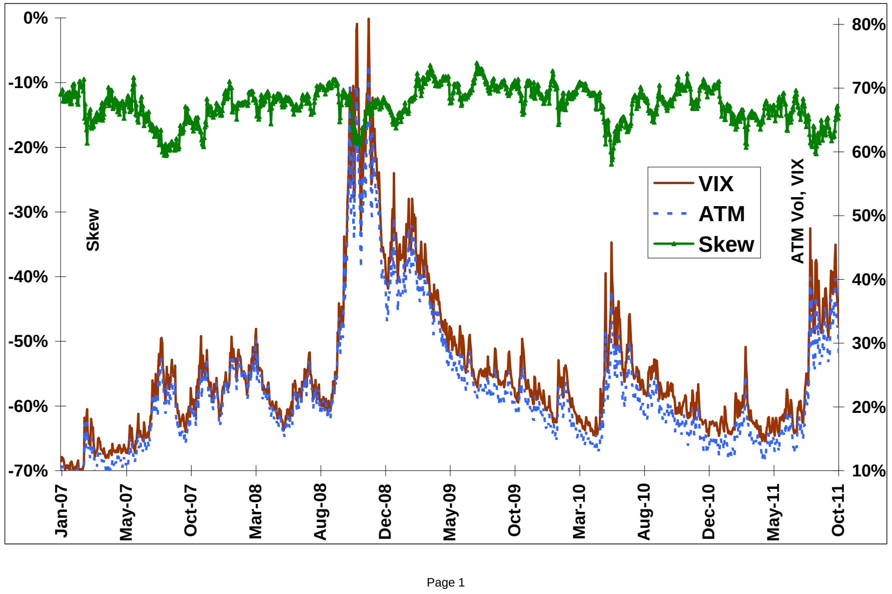
Find a dynamic model that can explain both the negative skew for equity options on the S&P500 index (SPX) and positive skew for options on the VIX



Implied vols of 1m options on the SPX and the VIX as functions of strike $K\%$ relative to 1m SPX forward and 1m VIX future, respectively

Motivation II. Dynamics

The VIX exhibits strong mean-reversion and jumps



Rights scale: Time series of the VIX and 1m ATM SPX implied volatility (ATM)
Left scale: 1m 110% – 90% skew

The VIX I. Definition

The VIX is a measure of the implied volatility of SPX options with maturity 30 days

Trading in VIX futures & options began in 2004 & 2006, respectively
Throughout 2004-2007, the average of the VIX is 14.66%
Throughout 2008-2011, the average of the VIX is 27.65%

Nowadays, VIX options are one of the most traded products on CBOE with the average daily about 10% of all traded contracts

Formally, the VIX at time t, denoted by $F(t)$, is the square root of the expectation of the quadratic variance $I(t,T)$ of log-return $\ln(S(T)/S(t))$

$$F(t) = \sqrt{\frac{1}{\tau_T}\mathbb{E}\left[I(t,t+\tau_T)\,|\,S(t)\right]}\ ,\ \ \tau_T = 30/365 \tag{2}$$

For a general non-linear pay-off function u:

$$U(t,T) = \mathbb{E}\left[u(F(T))\,|\,S(t)\right] = \mathbb{E}\left[u\left(\sqrt{\frac{1}{\tau_T}\mathbb{E}\left[I(T,T+\tau_T)\,|\,S(T)\right]}\right)\,|\,S(t)\right]$$

Valuation of the VIX options I. PDE method

Consider the augmented dynamics for $S(t)$ and $I(t)$:

$$\begin{aligned} dS(t) &= \sigma(S(t))dW(t),\ S(0) = S \\ dI(t) &= \left(\frac{\sigma(S(t))}{S(t)}\right)^2 dt,\ I(0) = 0 \end{aligned} \tag{3}$$

1) Solve backward problem for $V(t,S) = \mathbb{E}\left[I(T, T+\tau_T)\right]$ on time interval $[T, T+\tau_T]$:

$$V_t + \mathcal{L}V = -\left(\frac{\sigma(S)}{S}\right)^2\ ,\ V(T+\tau_T, S) = 0 \tag{4}$$

where $\mathcal{L}$ is the backward operator:

$$\mathcal{L} = \frac{1}{2}\sigma^2(S)\partial_{SS} \tag{5}$$

2) Solve backward problem for $U(t,S) = \mathbb{E}\left[u\left(F(T)\right)\right]$ on interval $[0,T]$:

$$U_t + \mathcal{L}U = 0\ ,\ U(T,S) = u\left(\sqrt{\frac{1}{\tau_T}V(T,S)}\right) \tag{6}$$

The problem is similar to valuing an option on a coupon-bearing bond

Valuation of the VIX options II. Enhancement

Solve the forward problem for the density function of S, $G(T, S')$, at time T

$$G_T - \widetilde{\mathcal{L}}G = 0 \ , \ G(0, S) = \delta(S' - S) \tag{7}$$

where $\widetilde{\mathcal{L}}$ is operator adjoint to $\mathcal{L}$:

$$\widetilde{\mathcal{L}}G = \frac{1}{2}\partial_{SS}\left(\sigma^2(S)G\right) \tag{8}$$

Then $U(T, S)$ is priced by

$$U(0, S) = \int_0^\infty u\left(\sqrt{\frac{1}{\tau_T}V(T, S')}\right)G(T, S')dS' \tag{9}$$

Generic implementation is done by means of finite-difference (FD) methods that allow to tackle the problem for arbitrary choice of $\sigma(S)$

When using (9) we only need to solve 2 PDE-s, backward and forward, to value VIX options with maturity T across different strikes

Stochastic volatility and jumps can be incorporated and treated by FD methods (see Sepp (2011) for a review)

Specific Models I

Dupire LV and LSV models are "**black boxes**" - for any admissible set of model parameters, they guarantee calibration to equity skew

Instead, I consider simple model specifications to analyze their calibration and implied VIX dynamics by fitting model parameters to:
1) at-the-money(ATM) implied volatility $\sigma_{\text{ATM, Eq}}$
2) equity skew defined by:

$$\text{Skew}_{\text{Eq}}(\alpha) \equiv \sigma_{\text{imp}}((1+\alpha)S) - \sigma_{\text{imp}}((1-\alpha)S)$$

where $\sigma_{\text{imp}}(K)$ is SPX implied vol function of strike K and $\alpha = 10\%$

As an output, I consider **VIX skew** defined by:

$$\text{Skew}_{\text{VIX}}(\alpha) \equiv \sigma_{\text{imp,VIX}}((1+\alpha)S) - \sigma_{\text{imp,VIX}}(S),$$

where $\sigma_{\text{imp}}(K)$ is VIX implied vol as function of strike K

Inputs on 31 October 2011 for 1m SPX and VIX options:
$S(0) = 1253.3$, $\sigma_{\text{ATM, Eq}} = 25\%$, $\text{Skew}_{\text{Eq}}(10\%) = -16\%$
$F(0) = 30\%$, $\sigma_{\text{ATM,VIX}} = 101\%$, $\text{Skew}_{\text{VIX}}(10\%) = 12\%$

Specific Models I. CEV model A: Basics

Consider the CEV process (Cox (1975)):

$$dS(t) = \sigma \left(\frac{S(t)}{S(0)}\right)^{\beta} dW(t) \tag{10}$$

We can derive an approximation for implied vol at $K = (1-\alpha)S$:

$$\sigma_{\text{imp}}((1-\alpha)S) = \sigma \left(\frac{S}{S(0)}\right)^{1-\beta} \left(1 + \frac{1}{2}\alpha(1-\beta) + O(\alpha^2)\right)$$

Thus

$$\sigma_{\text{ATM,CEV}}(S) \equiv \sigma \left(\frac{S}{S(0)}\right)^{1-\beta}, \ \text{Skew}_{\text{Eq,CEV}}(\alpha) \equiv -\sigma \left(\frac{S}{S(0)}\right)^{1-\beta} \alpha(1-\beta)$$

Given $\sigma_{\text{ATM,Eq}}$ and $\text{Skew}_{\text{Eq}}(\alpha)$:

$$\sigma = \sigma_{\text{ATM}} \left(\frac{S}{S(0)}\right)^{1-\beta}, \ \beta = 1 + \frac{\text{Skew}_{\text{Eq}}(\alpha)}{\alpha\sigma_{\text{ATM,Eq}}}$$

Specific Models II. CEV model C: Dynamics of the VIX
Approximate the VIX using:

$$F(t) \approx \frac{\sigma(S(t))}{S(t)} = \frac{\sigma\left(S(t)\right)^{\beta}}{S(t)} = \sigma\left(S(t)\right)^{\beta-1}$$

For the dynamics of $F(t)$:

$$dF(t) = (\beta - 1)F^2(t)dW(t) + \frac{1}{2}(\beta - 1)(\beta - 2)F^3(t)dt$$

Observations:
1), No mean-revertion

2), for $\beta < 1$, we have negative absolute correlation with the spot

Specific Models II. CEV model D: VIX implied volatility

We can show that approximately:

$$\sigma_{\text{imp,vix}}((1-\alpha)F) = -(\beta-1)F\left(1-\frac{1}{2}\alpha + O(\alpha^2)\right)$$

Thus:

$$\sigma_{\text{ATM,VIX,CEV}} = (1-\beta)F \ , \ \text{Skew}_{\text{VIX,CEV}} = (1-\beta)F\frac{1}{2}\alpha$$

Using implied parameters:

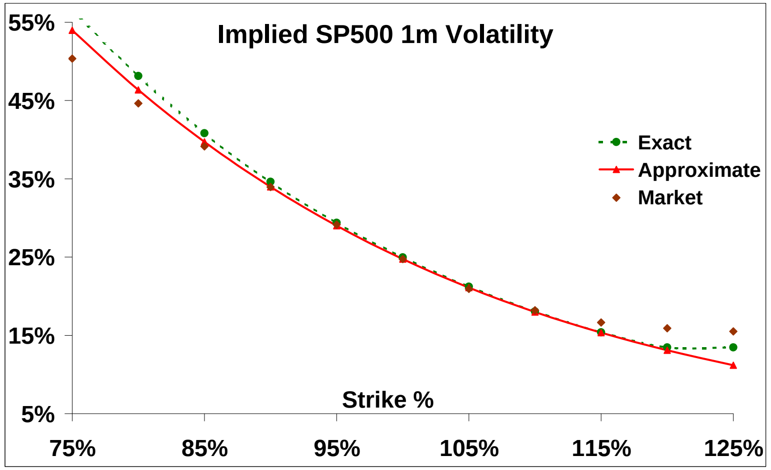
$$\sigma_{\text{ATM,VIX,CEV}} \approx -\frac{\text{Skew}_{\text{Eq}}(\alpha)}{\alpha} \ , \ \text{Skew}_{\text{VIX,CEV}} \approx -\frac{1}{2}\text{Skew}_{\text{Eq}}(\alpha)$$

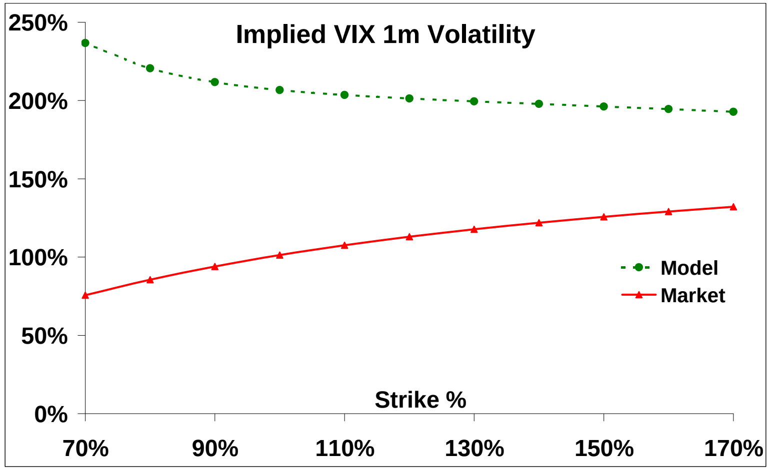
Both VIX ATM volatility and skew are proportional to the equity skew

Using $\text{Skew}_{\text{Eq}}(10\%) = -16\%$ we get:
$\sigma_{\text{ATM,VIX,CEV}} = 160\%$ (vs 100% actual)
$\text{Skew}_{\text{VIX,CEV}} = 8\%$ (vs 12% actual)

CEV (and LV in general) tend to overestimate VIX ATM vol and underestimate VIX skew

Specific Models II. CEV model E: Illustration





Left: SPXskew; Right: the VIX skew
Implied model parameters: $\sigma = 0.2474$, $\beta = -5.37$

Specific Models III. Jump-diffusion A: Basics
Merton (1973) jump-diffusion with discrete jump in log-return ν :

$$dS(t) = \sigma S(t) dW(t) + (e^{\nu} - 1)\, S(t)\, [dN(t) - \lambda dt] \tag{11}$$

where $N(t)$ is Poisson with intensity λ

We can show that to the leading order for small time-to-maturity:

$$\sigma_{\mathsf{imp}}(K) \approx \sigma - \frac{\lambda \nu}{\sigma} \ln(S/K)$$

The short-term skew in JD is linear in jump size and its intensity

Specific Models III. Jump-diffusion B: Equity implied volatility

We have for small time-to-maturity:

$$\sigma_{\text{imp}}(S) \approx \sigma, \ \text{Skew}_{\text{Eq}}(\alpha) \approx 2\alpha \frac{\lambda\nu}{\sigma}$$

JD model is overdetermined so consider expected quadratic variance:

$$\upsilon = \sigma^2 + \lambda\nu^2$$

Introduce weight w_{jd}, $0 < w_{jd} < 1$, as proportion of variance contributed by jumps (a parameter for calibration) and take $\upsilon = \sigma^2_{\text{ATM}}$:

$$1 = \frac{\sigma^2}{\sigma^2_{\text{ATM}}} + \frac{\lambda\nu^2}{\sigma^2_{\text{ATM}}} \equiv (1 - w_{jd}) + w_{jd}$$

Thus, obtain equation to imply λ by:

$$\frac{\lambda\nu^2}{\sigma^2_{\text{ATM}}} = w_{jd} \Rightarrow \lambda = w_{jd} \frac{\sigma^2_{\text{ATM}}}{\nu^2}$$

Accordingly:

$$\nu = \frac{2\alpha w_{jd}\sigma_{\text{ATM}}}{\text{Skew}_{\text{Eq}}(\alpha)} \ , \ \lambda = \frac{(\text{Skew}_{\text{Eq}})^2}{4\alpha^2 w_{jd}} \ , \ \sigma \approx \sigma_{\text{ATM}}$$

Specific Models III. Jump-diffusion E: VIX dynamics

The quadratic variance in the jump-diffusion model is driven by:

$$dI(t) = \sigma^2 dt + \nu^2 dN(t)$$

The variance swap

$$V(t) = \frac{1}{T}\mathbb{E}\left[\int_0^T I(t')dt'\right] = \sigma^2 + \lambda\nu^2$$

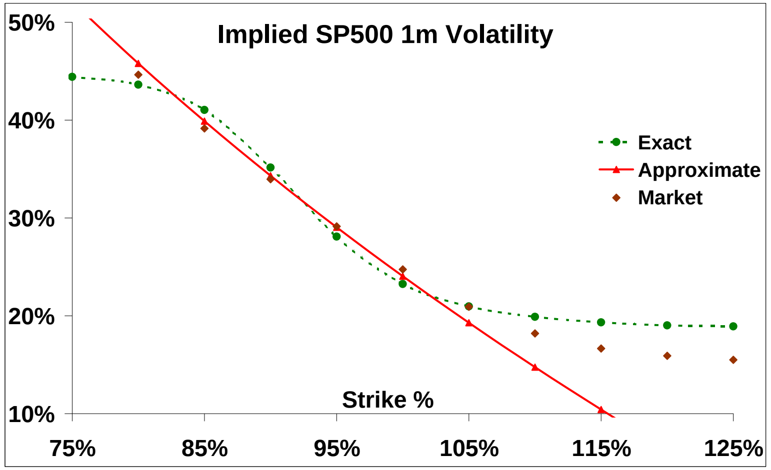
turns out to be a deterministic function

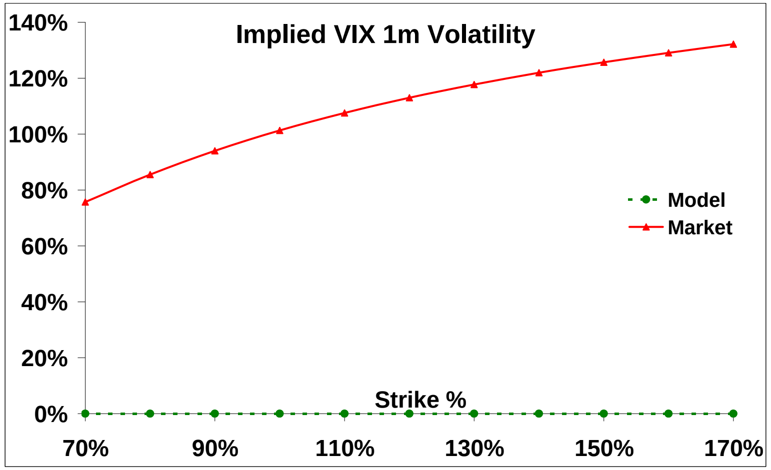
Thus, the model value of the VIX is constant and the implied volatility of the VIX is zero - even though the JD model generates the equity skew!

Thus property is shared by all space-homogeneous jump-diffusions and Levy processes!

Nevertheless, jumps are needed for calibration of VIX skew

Specific Models III. Jump-diffusion F: Illustration





Left: SPX skew; Right: VIX skew
Implied model parameters: $w_{jd} = 0.8$, $\lambda = 0.7762$, $\nu = -0.2512$, $\sigma = 0.1769$

Specific Models IV. CEV with jumps A: Basics

Consider process:

$$dS(t) = \sigma \left(\frac{S(t)}{S(0)} \right)^{\beta} dW(t) + (e^{\nu} - 1) \, S(t) \, [dN(t) - \lambda dt] \tag{12}$$

It turns out that the impact on skew from local volatility and jumps is roughly linear and additive

Thus, specify the weight for the skew explained by jumps, w_{jd}, and CEV local volatility w_{lv}, $w_{lv} = 1 - w_{jd}$

Using obtained results for CEV and JD models:

$$\beta = 1 + \frac{w_{lv} \mathrm{Skew}_{\mathrm{Eq}}(\alpha)}{\alpha \sigma_{\mathrm{ATM}}} \, , \; \nu = \frac{2 \alpha \sigma_{\mathrm{ATM}}}{\mathrm{Skew}_{\mathrm{Eq}}(\alpha)} \, , \; \lambda = \frac{w_{jd} (\mathrm{Skew}_{\mathrm{Eq}})^2}{4 \alpha^2}$$

$$\sigma = \sigma_{\mathrm{ATM}} \left(\frac{S(t)}{S(0)} \right)^{1-\beta}$$

Specific Models IV. CEV with jumps B: The VIX
The variance swap can be approximated by:

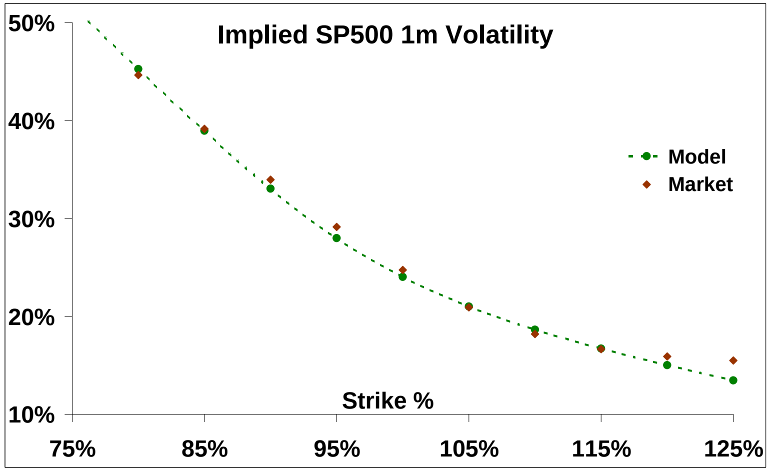
$$V(0,T) = \frac{1}{T}\mathbb{E}\left[\int_0^T \left[\frac{dS(t')}{S(t')}\right]^2 dt'\right] \approx \sigma^2 \left(\frac{S(t)}{S(0)}\right)^{2\beta-2} + \lambda\nu^2,$$

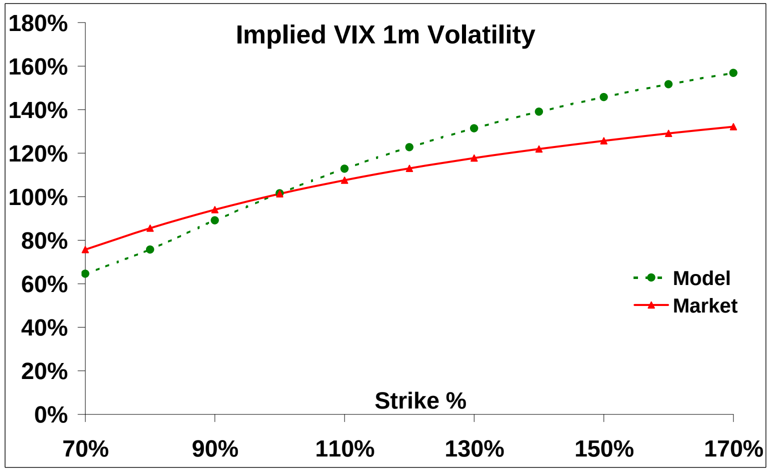
The VIX is approximated using:

$$F(t) \approx \sqrt{\sigma^2 S^{2\beta-2}(t) + \lambda\nu^2},$$

Observation: in CEVJD, β becomes smaller (in absolute terms) thus the implied vol and skew of the VIX decrease

Specific Models V. CEV with jumps C: Illustration





Left: S&P500 skew; Right: the VIX skew
Implied model parameters: $w_{jd} = 0.4$ (fitted to match the VIX ATM volatility), $\lambda = 0.2484$, $\nu = -0.3140$, $\sigma = 0.2192$, $\beta = -3.3141$

Specific Models V. Stochastic volatility A: Basics
Consider exponential OU stochastic volatility (Scott (1987) SV model, Stein-Stein (1991) SV model is linear in $Y(t)$):

$$\begin{aligned} dS(t) &= \sigma S(t) e^{Y(t)} dW^{(0)}(t) \\ dY(t) &= -\kappa Y(t) + \varepsilon dW^{(1)}(t),\ Y(0) = 0 \end{aligned} \tag{13}$$

with $dW^{(0)}(t) dW^{(1)}(t) = \rho dt$
κ is mean-reversion speed
ε is volatility-of-volatility

For equity skew, we can show approximately:

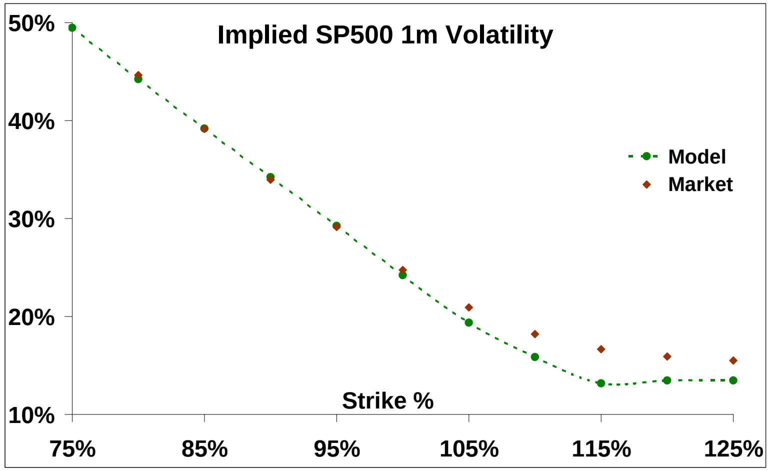
$$\sigma_{\text{ATM}} \approx \sigma\ ,\ \text{Skew}_{\text{Eq}}(\alpha) \approx \alpha \frac{\rho\varepsilon}{4\sigma} \Rightarrow \rho = \frac{4\sigma \text{Skew}_{\text{Eq}}(\alpha)}{\alpha\varepsilon}$$

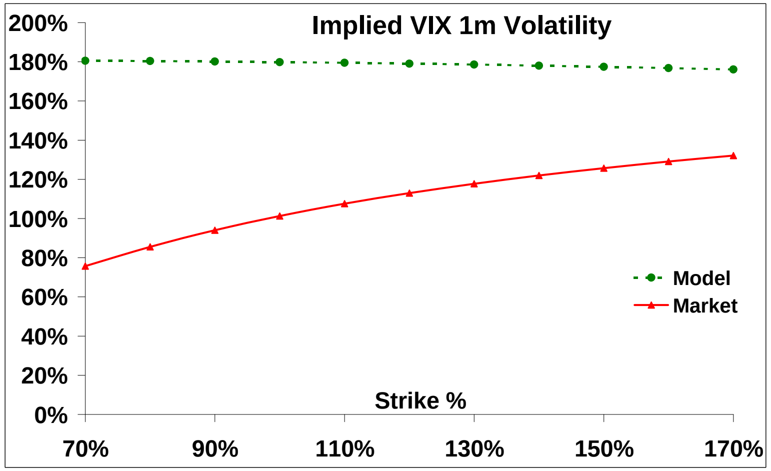
For the VIX implied volatility, we can show approximately:

$$\sigma_{imp,vix} = \frac{\sigma\varepsilon}{F^2(t)}\ ,\ \text{Skew}_{vix}(\alpha) = -\frac{\sigma\varepsilon}{2F^2(t)}\alpha$$

A) The ATM implied vol and the skew are proportional to $\sigma\varepsilon$;
B) The VIX skew is negative

Specific Models V. Stochastic volatility D: Illustration





Left: SPX skew; Right: the VIX skew
Implied model parameters: $\kappa = 4.48$, $\varepsilon = 2.69$, $\rho = -0.85$, $\sigma = 0.2194$,

Specific Models. Summary

We have considered several models with conclusions:

Model	Equity Skew	VIX Skew	Mean-revertion
CEV	Yes	Weak	No
Jump-Diffusion	Yes	No	No
CEV Jump-Diffusion	Yes	Yes (too steep)	No
Stochastic Volatility	Yes	No	Yes

To model the VIX skew, we need three components:
Local volatility (for equity and VIX skew)
Stochastic volatility (mean-reverting feature of the Vix)
Jumps in price and volatility (steeper equity and the VIX skew)

<u>Two viable contenders</u>:
1) SV model with jumps (space-inhomogeneous so some analytical solutions possible - Sepp (2008))
2) LSV model with jumps, for example, CEV or non-parametric (valuation by means of FD methods)

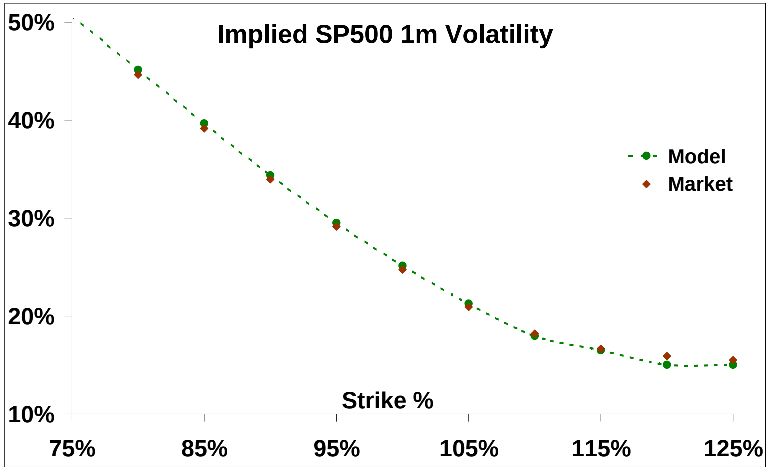
Specific Models VI. SV+Jump-Diffusion: Dynamics
Consider generic LSV model with jumps:

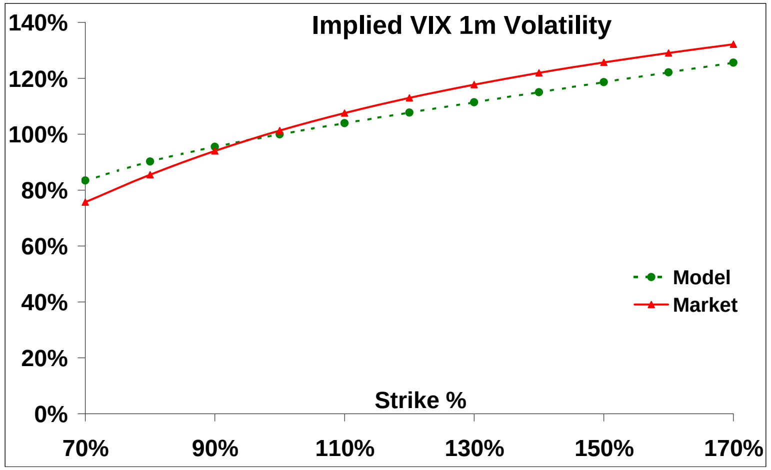
$$dS(t) = \sigma S(t) e^{Y(t)} dW^{(0)}(t) + S(t) \left[\left(e^{\nu} - 1 \right) dN(t) - \lambda \overline{\nu} dt \right]$$
$$dY(t) = -\kappa Y(t) + \varepsilon dW^{(1)}(t) + J^{v} dN(t)$$
$$dI(t) = \sigma^2 e^{2Y(t)} dt + \left(J^{s} \right)^2 dN(t)$$

with $dW^{(0)}(t) dW^{(1)}(t) = \rho dt$
$N(t)$ is Poisson process with intensity λ
Jumps in $S(t)$ and $Y(t)$ are **simultaneous and discrete** with magnitudes $\nu < 0$ and $\eta > 0$

Here:
$\sigma = \sigma(t)$ - SV with jumps
$\sigma = \sigma(t) \left(\frac{S(t)}{S(0)} \right)^{\beta}$ - CEV+SV with jumps
$\sigma = \sigma_{(\text{loc,svj})}(t, S(t))$ - LSV with jumps (Lipton (2002))

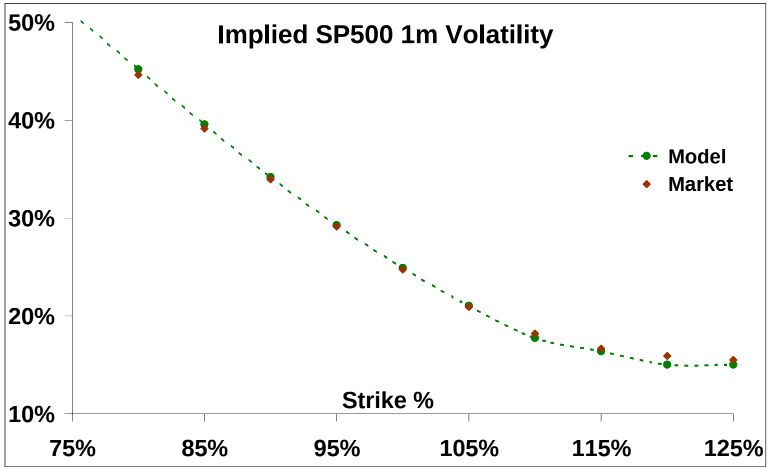
Specific Models VI. SV+Jump-Diffusion: Illustration

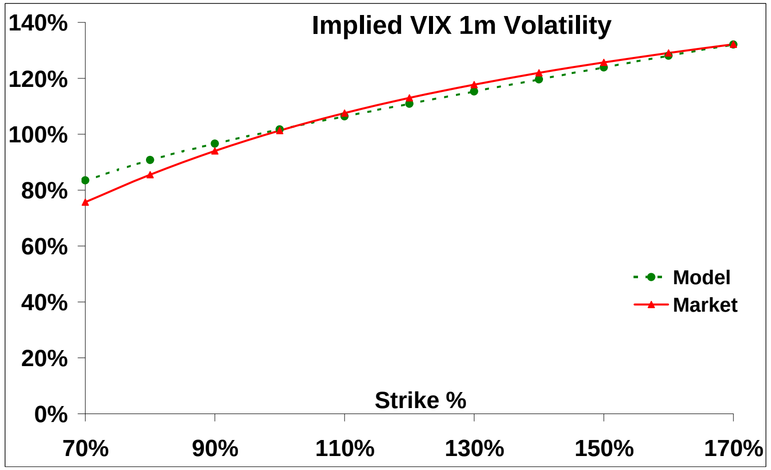




Left: SPX skew; Right: the VIX skew
Implied model parameters: $w_{sv} = 0.65$, $w_{jd} = 0.35$ (fitted to match the VIX ATM volatility), $\lambda = 0.2173$, $\nu = -0.314$, $\rho = -0.85$, $\sigma = 0.2227$, $\kappa = 4.48$, $\varepsilon = 1.7485$ $(= w_{sv}\widehat{\varepsilon})$, $\eta = 1.30$ (fitted to match to fit the VIX skew)

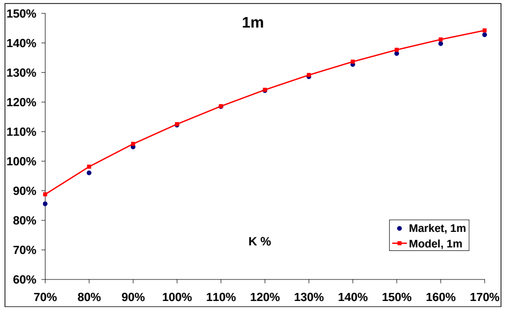
Specific Models VI. CEV+SV+Jump-Diffusion: Illustration

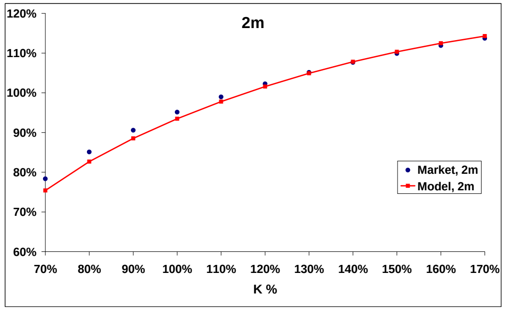


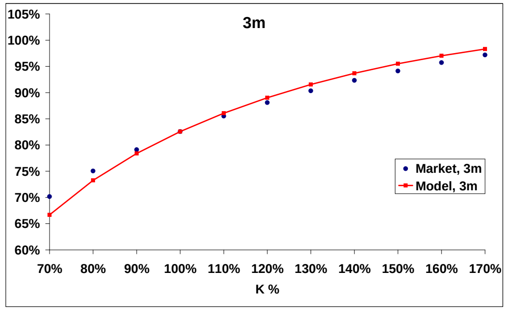


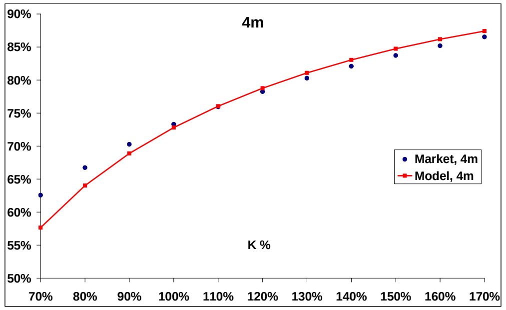
Left: S&P500 skew; Right: the VIX skew
Implied model parameters: $w_{jd} = 0.35$, $w_{cev} = 0.05$, $w_{sv} = (1 - w_{cev})(1 - w_v) = 0.6175$ (fitted to match the VIX ATM volatility), $\beta = 0.77$, $\lambda = 0.2173$, $\nu = -0.314$, $\rho = -0.85$, $\sigma = 0.2227$, $\kappa = 4.48$, $\varepsilon = 1.6611$ $(= w_{sv}\widehat{\varepsilon})$, $\eta = 1.30$ (fitted to match to fit the VIX skew)

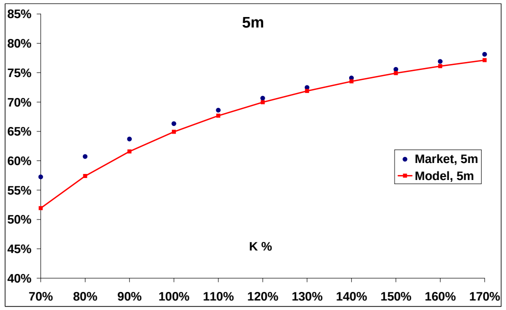
Specific Models VI. Non-parametric LSV+Jump-Diffusion

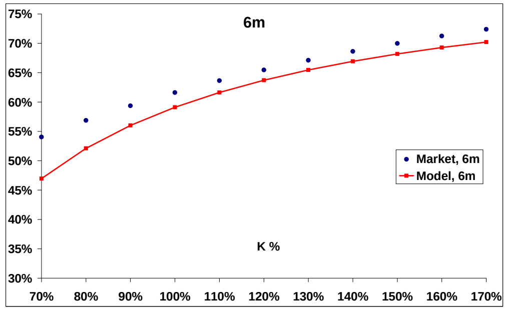












LSV model fit to VIX implied volatilities (the model is consistent with the S&P500 implied volatilities by construction)
Implied model parameters: $w_{jd} = 0.35$, $w_{lv} = 0.05$, $w_{sv} = 0.6175$, $\lambda = 0.2173$, $\nu = -0.314$, $\rho = -0.80$, $\kappa = 4.48$, $\varepsilon = 1.6611$, $\eta = 1.30$

Conclusions
For calibration of LSV model to SPX and VIX skews, a careful choice between the stochastic volatility, local volatility and jumps is necessary:

1) Local volatility: weight w_{lv} - calibration parameter (typically small)

2) Price jumps: weight w_{jd} - calibration parameter (fitted to VIX ATM volatility), λ and ν determined from given equity skew

3) Volatility jumps: η - calibration parameter (fitted to VIX skew)

4) SV parameters: SV weight $w_{sv} = 1 - w_{lv} - w_{jd}$ - calibration variables, κ and ε from historical (implied) data

VIX options give extra information for calibration of LSV models

Part II. Joint calibration of medium- and long-term SPX skews using two-factor stochastic volatility model

Observation: One-factor stochastic volatility model with time-independent parameters can only calibrate to
A) either medium-term skews (up to 1 year) using large mean-reversion and vol-of-vol
B) or long-term skews (from 1 year) using low mean-reversion and vol-of-vol

Idea: introduce a two-factor stochastic volatility model with one factor for short-term skew and second factor for long-term skew

Challenge: find a proper weight (time-dependent) for the two factors

Bergomi (2005) assumes constant weights - I found it difficult to calibrate his model with constant weights

Skew decay I

Consider 110%−90% market skew, $\text{Skew}(T_n)$, for maturities $\{1m, 2m, ..., 60m\}$

Compute skew decay rate: $R_{\text{market}}(T_n) = \frac{\text{Skew}(T_n)}{\text{Skew}(T_{n-1})}$, $n = 2, ..., 60$

In addition, consider the following test functions:

$$F_\alpha(T_n) = (T_n)^{-\alpha}$$

with decay rate $R_\alpha(T_n) = \frac{F_\alpha(T_n)}{F_\alpha(T_{n-1})} = \left(\frac{T_n}{T_{n-1}}\right)^{-\alpha}$

We observe:

1) for small T, the skew decay $\alpha \approx 0.25$

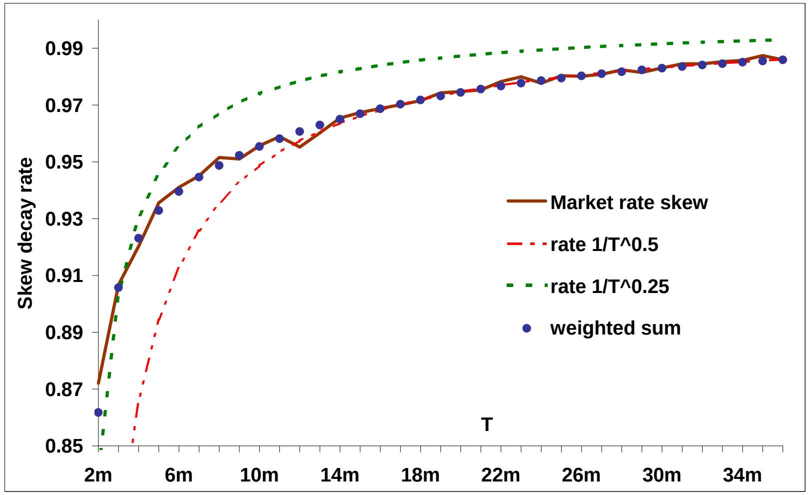
2) for large T, the skew decay $\alpha \approx 0.50$

Experiment - compute weighted average:

$$F_{\alpha_1,\alpha_2}(T_n) = \omega(T_n)T_n^{-\alpha_1} + (1 - \omega(T_n))T_n^{-\alpha_2}\ ,\ \omega(t) = e^{-\kappa t}\ ,\ \kappa = 2$$

with skew decay rate: $R_{\alpha_1,\alpha_2}(T_n) = \frac{F_{\alpha_1,\alpha_2}(T_n)}{F_{\alpha_1,\alpha_2}(T_{n-1})}$

Skew decay II



For short maturities, the skew is proportional to $T^{-0.25}$
For longer maturities, the skew is proportional to $T^{-0.50}$

The decay of the market skew is reproduced using "weighted" skew:

$$F_{0.25,0.50}(T_n) = \omega(T_n)T_n^{-0.25} + (1 - \omega(T_n))T_n^{-0.5} \;,\; \omega(t) = e^{-\kappa t} \;,\; \kappa = 2$$

Two-factor SV model I
Start with **one-factor SV dynamics** for $S(t)$ and SV factor $Y(t)$:

$$dS(t)/S(t) = \mu(t)dt + \sigma \exp\{Y(t)\}\, dW^{(0)}$$
$$dY(t) = -\kappa Y(t)dt + \varepsilon dW^{(1)},\ dW^{(0)}dW^{(1)} = \rho$$

Then extend to **two-factor SV dynamics** :

$$dS(t)/S(t) = \mu(t)dt + \sigma \exp\{\omega(t)Y_1(t) + (1-\omega(t))Y_2(t)\}\, dW^{(0)}$$
$$dY_1(t) = -\kappa_1 Y_1(t)dt + \varepsilon_1 dW^{(1)}\ ,\ dY_2(t) = -\kappa_2 Y_2(t)dt + \varepsilon_2 dW^{(2)}$$

$dW^{(0)}dW^{(1)} = \rho_{01}$, $dW^{(0)}dW^{(2)} = \rho_{02}$, $dW^{(1)}dW^{(2)} = \rho_{12} \equiv \rho_{01}\rho_{02}$
Here:
$Y_1(t)$ - SV factor for **short-term volatility**;
$Y_2(t)$ - SV factor for **long-term volatility**;
$\omega(t)$, $0 < \omega(t) < 1$, - **weight** between short-term and long-term factor:

$$\omega(t) = \exp\left\{-\frac{1}{2}(\kappa_1 + \kappa_2)\, t\right\}$$

We expect: $\kappa_1 >> \kappa_2$, $\varepsilon_1 >> \varepsilon_2$, $|\rho_{01}| > |\rho_{02}|$

Two-factor SV model II. Calibration
The model is implemented using a 3-d PDE solver with a predictor-corrector
Typically, we use $N_1 = 100$ per year in time dimension, $N_2 = 200$ in spot dimension, $N_3 = N_4 = 25$ in volatility dimensions

For calibration, we solve the forward PDE to compute the density of the underlying price and value European calls and puts at once (takes about 1-minute to calibrate to ATM volatility up to 5 years)

To simplify the calibration we consider two quantities at given maturity objects: the ATM volatility and $90\% - 110\%$ skew

1) calibrate two sets of parameters $\{\kappa, \varepsilon, \rho\}$ for 1-factor SV model:
the first one - to short term skew (up to 1 year)
the second one - to long-term skew (from 1y to 5y)
2) Use these parameters for the 2-factor SV model with weight $\omega(t)$

For all three models, piece-wise constant volatilities $\{\sigma(T_n)\}$ are calibrated so that the model matches given market ATM volatility at maturity times $\{T_n\}$

Two-factor SV model III. Illustration of calibration

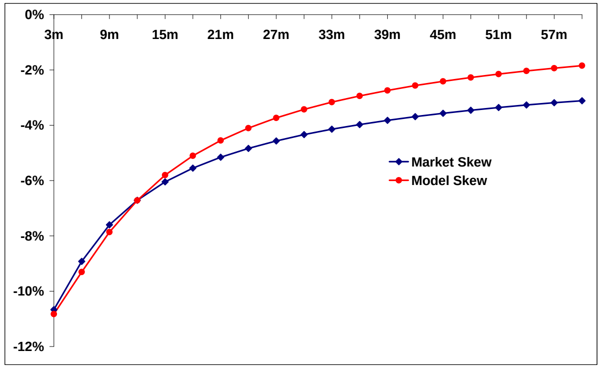
Calibrated parameters for SPX (top) and STOXX 50 (bottom)

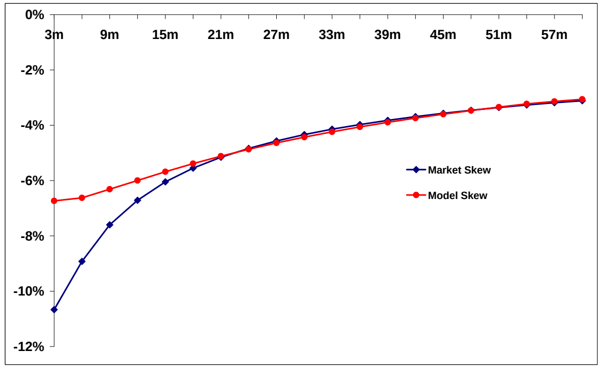
	Short-term	Long-term	2-f model
κ	3.50	0.65	
ε	1.95	0.95	
ρ	-0.85	-0.80	
ρ_{12}			0.68
$\frac{1}{2}(\kappa_1 + \kappa_2)$			2.08

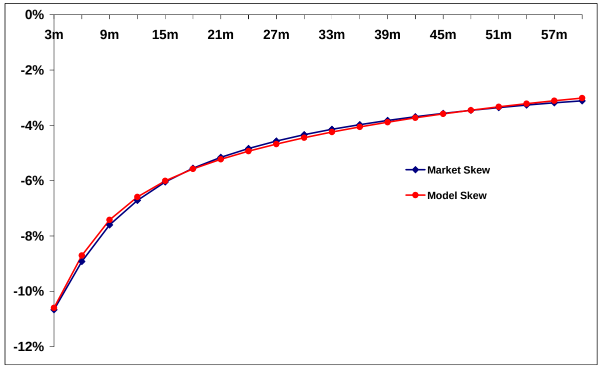
	Short-term	Long-term	2-f model
κ	2.40	0.55	
ε	1.40	0.70	
ρ	-0.85	-0.80	
ρ_{12}			0.68
$\frac{1}{2}(\kappa_1 + \kappa_2)$			1.48

Two-factor SV model IV. Calibration to SPX

Term structure of skews for 1-f SV model calibrated up to 1y skews, 1-f SV model calibrated from 1y to 5y skews, 2-factor SV model

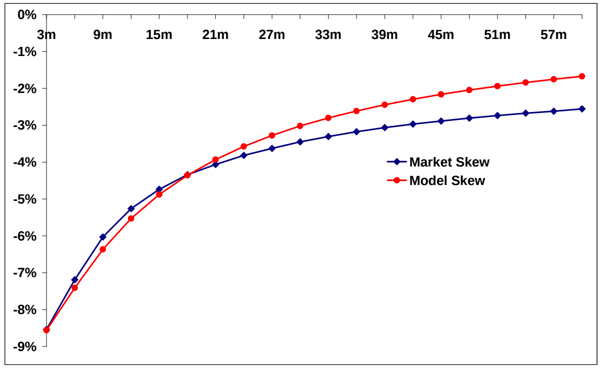


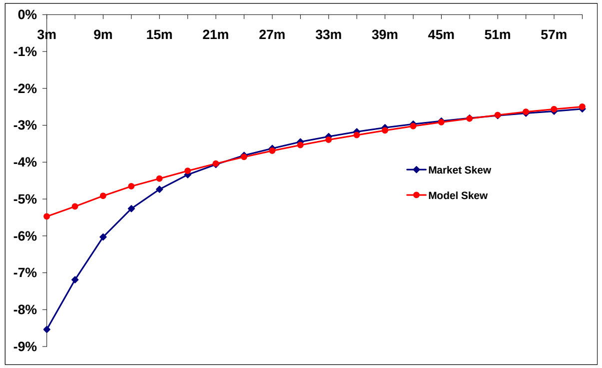




Two-factor SV model V. Calibration to STOXX 50

Term structure of skews for 1-f SV model calibrated up to 1y skews, 1-f SV model calibrated from 1y to 5y skews, 2-factor SV model





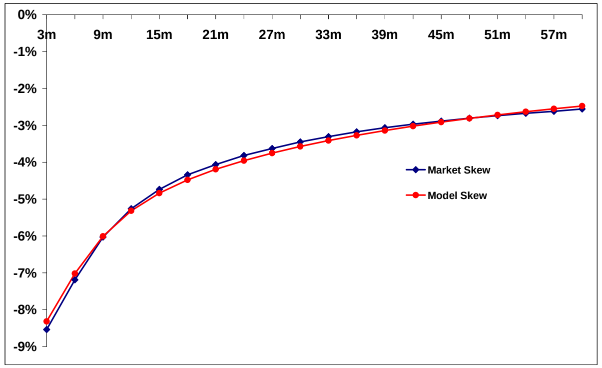
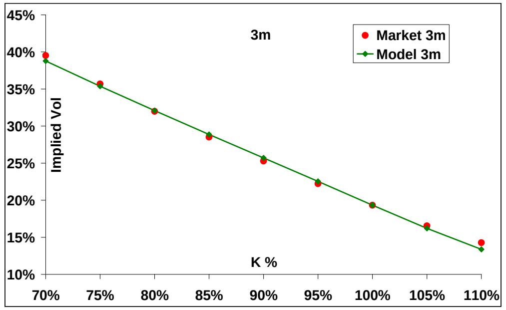
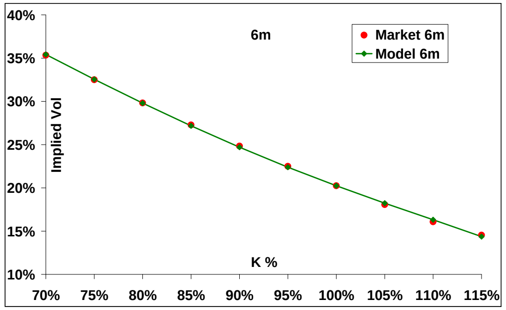


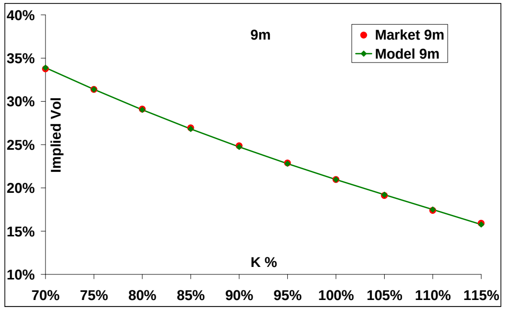
Illustration VI. Model implied vol across strikes for SPX

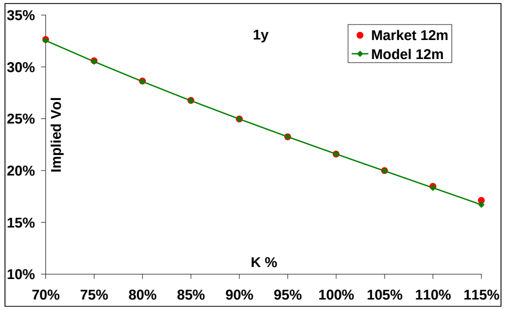
A) 2-factor SV model with piecewise-constant volatility fits the term-structure of market skews, unlike 1-factor SV model

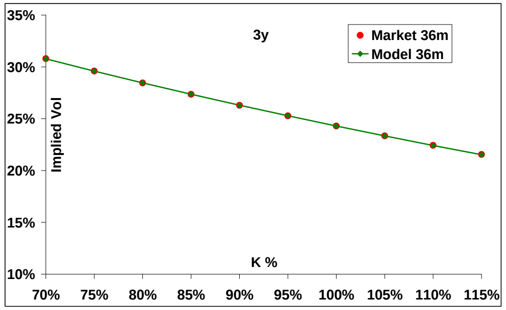
B) Small discrepancies in model and market implied volatilities across strikes are eliminated by introducing a parametric local volatility with extra parameter to match the skew across strikes











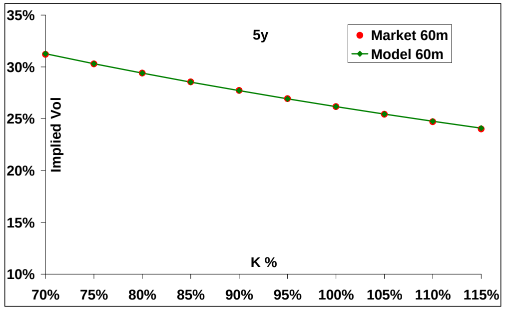
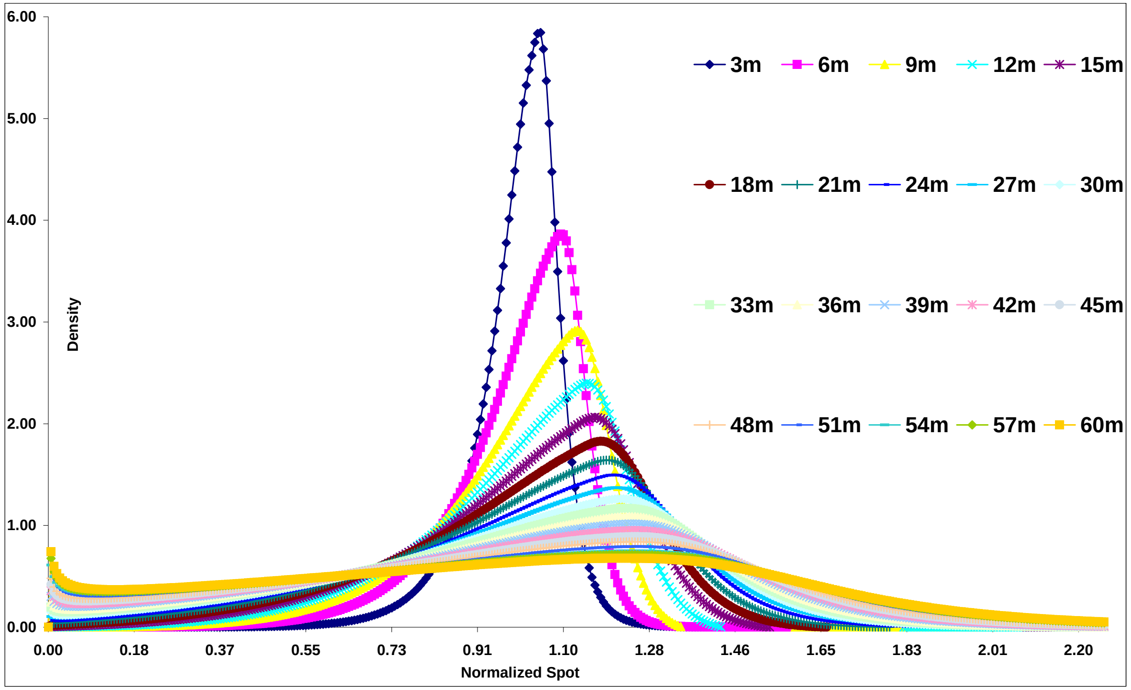
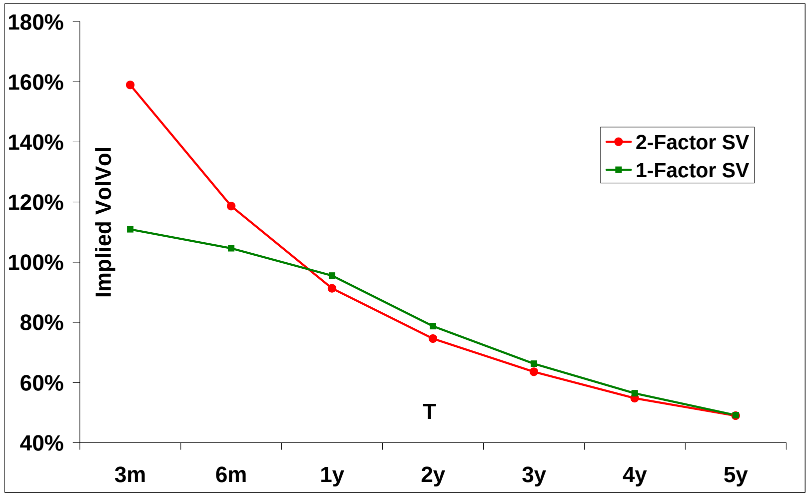


Illustration VII. Model implied density for SPX



Observe heavy tails for longer-maturities with concentration around $S = 0$ (zero is assumed to be an absorbing barrier for my PDE solver)

Illustration VIII. The term structure of implied volatility-of-volatility



The term structure of implied volatility-of-volatility (implied log-normal vol for ATM option on the realized variance) in 1- and 2-factor SV model

Conclusions

I have presented:

1) Local stochastic volatility model with jumps for modeling of short-term skews and illustrated its calibration to VIX options skews

I have shown that jumps are needed to fit the model to both SPX and VIX skews

2) 2-factor stochastic volatility model for modeling of medium- and long-dated skews

I have shown that only two-factor stochastic volatility model can fit both medium- and long-dated skews

For both models, I have tried to emphasize an intuitive approach to model calibration and avoid using "blind" non-linear calibrations

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During my work I benefited from interactions with:

Alex Lipton on his universal local stochastic volatility with jumps (Lipton (2002))

Hassan El Hady, Charlie Wang and other members of BAML Global Quantitative Analytics

The opinions and views expressed in this presentation are those of the author alone and do not necessarily reflect the views and policies of Bank of America Merrill Lynch

Thank you for your attention!

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