An Approximate Distribution of Delta-Hedging Errors in a Jump-Diffusion Model with Discrete Trading and Transaction Costs

Artur Sepp

Bank of America Merrill Lynch

Artur.Sepp@baml.com

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Motivation

1) Analyse the distribution of the profit&loss (P&L) of delta-hedging strategy for vanilla options in Black-Scholes-Merton (BSM) model and an extension of the Merton jump-diffusion (JDM) model assuming discrete trading and transaction costs

2) Examine the connection between the realized variance and the realized P&L

3) Find approximate solutions for the P&L volatility and the expected total transaction costs;
Apply the mean-variance analysis to find the trade-off between the costs and P&L variance given hedger's risk tolerance

4) Consider hedging strategies to minimize the jump risk

References

Theoretical and practical details for my presentation can be found in:

1) Sepp, A. (2012) An Approximate Distribution of Delta-Hedging Errors in a Jump-Diffusion Model with Discrete Trading and Transaction Costs, Quantitative Finance, 12(7), 1119-1141
`http://ssrn.com/abstract=1360472`

2) Sepp, A. (2013) When You Hedge Discretely: Optimization of Sharpe Ratio for Delta-Hedging Strategy under Discrete Hedging and Transaction Costs, Journal of Investment Strategies, 3(1), 19-59
`http://ssrn.com/abstract=1865998`

Literature

Discrete trading and transaction costs: Leland (1985), Boyle-Vorst (1992), Avellaneda-Paras (1994), Hoggard-Whalley-Wilmott (1994), Toft (1996), Derman (1999), Bertsimas-Kogan-Lo (2000)

Optimal delta-hedging: Hodges-Neuberger (1989), Edirsinghe-Naik-Uppal (1993), Davis-Panas-Zariphopoulou (1993), Clewlow-Hodges (1997), Zakamouline (2006), Albanese-Tompaidis (2008)

Robustness of delta-hedging: Figlewski (1989) , Crouhy-Galai (1995), Karoui-Jeanblanc-Shreve (1998), Crépey (2004), Carr (2005), Andersen-Andreasen (2000), Kennedy-Forsyth-Vetzal (2009)

Textbooks: Natenberg (1994), Taleb (1997), Lipton (2001), Rebonato (2004), Cont-Tankov (2004), Wilmott (2006), Sinclair (2008)

Plan of the presentation

1) The P&L in the BSM model with trading in continuous and discrete time

2) Extended Merton jump-diffusion model

3) Analytic approximation for the probability density function (PDF) of the P&L

4) Illustrations

BSM model

Asset price dynamics under the BSM:

$$dS(t) = S(t)\sigma_a dW(t),\ S(t_0) = S$$

Option value $U^{(\sigma)}(t,S)$, valued with volatility σ, solves the BSM PDE:

$$U_t^{(\sigma)} + \frac{1}{2}\sigma^2 S^2 U_{SS}^{(\sigma)} = 0,\ U^{(\sigma)}(T,S) = u(S)$$

The delta-hedging portfolio with option delta $\Delta^{(\sigma)}(t,S) \equiv U_S^{(\sigma)}(t,S)$:

$$\Pi^{(\sigma_h,\sigma_i)}(t,S) = S(t)\Delta^{(\sigma_h)}(t,S) - U^{(\sigma_i)}(t,S)$$

In general, we have three volatility parameters:
σ_i - for computing option value (implied volatility),
σ_h - for computing option delta (hedging volatility),
σ_a - for the "assumed" dynamics of the spot $S(t)$ (realized volatility).

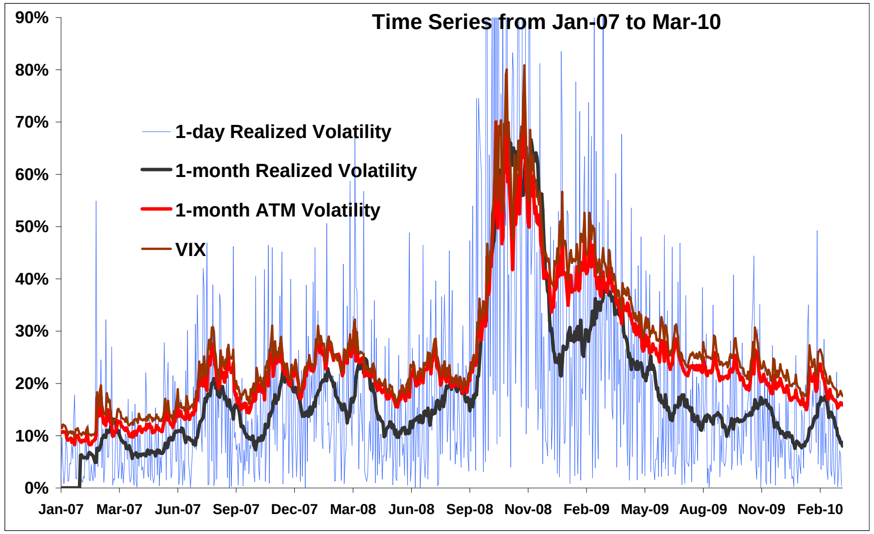
Why different?:
σ_i - market "consensus"
σ_h - proprietary view on option delta, risk management limits
σ_a - "guess estimated" by proprietary and subjective methods

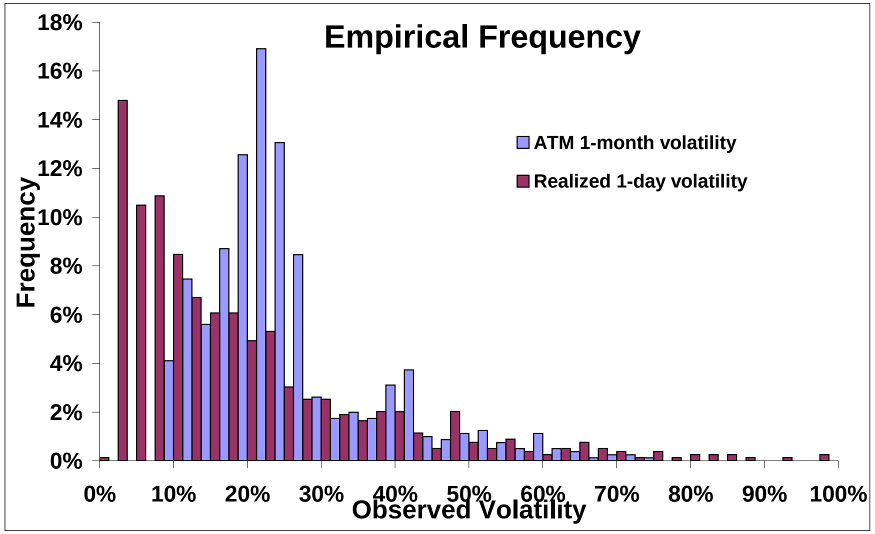
Illustration using the S&500 data

σ_i is illustrated by the VIX - market "consensus" about the one-month fair volatility ("model-independent")

σ_h is illustrated by the at-the-money one-month implied volatility

σ_a is illustrated by the one-month realized volatility





In general, the implied volatility is traded at the premium to the realized average volatility

But, over short-term periods, the realized volatility can be "jumpy", which causes significant risks for volatility sellers

Impact on the P&L when using different volatilities
One-period P&L at time $t+\delta t$ and spot $S+\delta S$ with fixed $\Delta^{(\sigma_h)}(t,S)$:

$$\begin{aligned}\delta\Pi^{(\sigma_h,\sigma_i)}(t) &\equiv \Pi^{(\sigma_h,\sigma_i)}(t+\delta t, S+\delta S) - \Pi^{(\sigma_h,\sigma_i)}(t,S)\\ &= \delta S \Delta^{(\sigma_h)}(t,S) - \left(U^{(\sigma_i)}(t+\delta t, S+\delta S) - U^{(\sigma_i)}(t,S)\right)\end{aligned}$$

Apply Taylor expansion and use BSM PDE:

$$\delta\Pi^{(\sigma_h,\sigma_i)}(t) \approx \delta S\left(\Delta^{(\sigma_h)}(t,S) - \Delta^{(\sigma_i)}(t,S)\right) + \delta t\left(\sigma_i^2 - \Sigma\right)\Gamma^{(\sigma_i)}(t,S)$$

$\Gamma^{(\sigma)}(t,S) = \frac{1}{2}S^2(t)U_{SS}^{(\sigma)}(t,S)$ is cash-gamma

$\Sigma = \frac{1}{\delta t}\left(\frac{\delta S}{S}\right)^2$ is the realized variance
In the limit: $\Sigma \to \sigma_a^2$ as $\delta t \to 0$

The stochastic term in the P&L is caused by mis-match between the implied and hedging volatilities
The deterministic term in the P&L is caused by mis-match between the implied and actual volatilities

Analysis of the total P&L, $P(T)$:

$$P(T) = \lim_{N\to\infty} \sum_{n=1}^{N} \delta\Pi^{(\sigma_h,\sigma_i)}(t_n)$$

1) BSM case: $\sigma_h = \sigma_i = \sigma_a$ thus $\delta\Pi^{(\sigma_a,\sigma_a)}(t) = 0, \; P(T) = 0$
No pain - no gain

2) Delta-hedge at implied volatility: $\sigma_h = \sigma_i$ thus

$$\delta\Pi^{(\sigma_i,\sigma_i)}(t) = \delta t \left(\sigma_i^2 - \sigma_a^2\right) \Gamma^{(\sigma_i)}(t,S),$$
$$P(T) = \left(\sigma_i^2 - \sigma_a^2\right) \int_{t_0}^{T} \Gamma^{(\sigma_i)}(t,S)dt$$

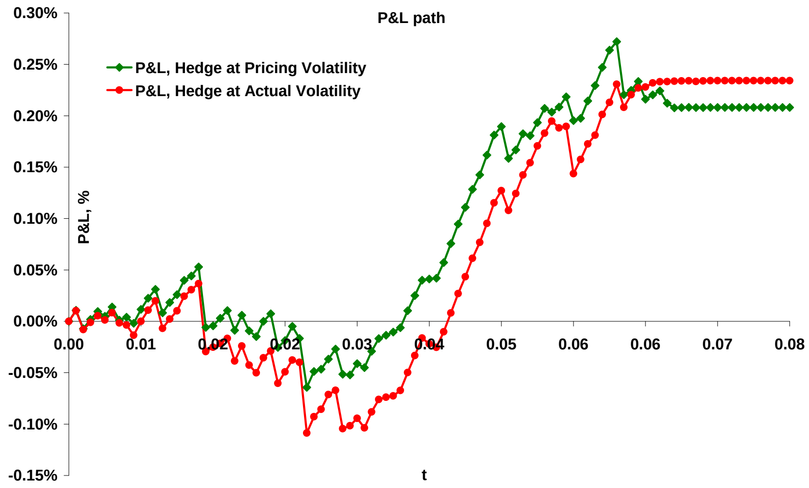
No pain across the path - unpredictable gain at expiry

3) Delta-hedge at the asset volatility: $\sigma_h = \sigma_a$, thus

$$\delta\Pi^{(\sigma_a,\sigma_i)}(t) = \delta S \left(\Delta^{(\sigma_a)}(t,S) - \Delta^{(\sigma_i)}(t,S)\right),$$
$$P(T) = U^{(\sigma_i)}(t_0,S) - U^{(\sigma_a)}(t_0,S)$$

Pain across the path - predictable gain at expiry

Illustration of a P&L path



$\sigma_i = 23\%$, $\sigma_a = 21\%$, $K = S = 1.0$, $T = 0.08$ (one month), $N = 100$ (hourly hedging)

The common practice is to hedge at implied volatility so no proprietary view on option delta is allowed

In the sequel, we consider the case with hedging at implied volatility

The P&L in the discrete time

Assume that the number of trades N is finite and the delta-hedging is rebalanced at times $\{t_n\}$, $n = 0, 1, ..., N$, with intervals $\delta t_n = t_n - t_{n-1}$
Recall that, under the continuous trading, the final P&L is

$$P(T) = \left(\sigma_i^2 - \sigma_a^2\right) \int_{t_0}^{T} \Gamma^{(\sigma_i)}(t, S) dt$$

Is it correct?:

$$P(T; \{t_n\}) = \left(\sigma_i^2 - \sigma_a^2\right) \sum_{n=1}^{N} \Gamma^{(\sigma_i)}(t_{n-1}, S) \delta t_n$$

The correct form is:

$$P(T; \{t_n\}) = \sum_{n=1}^{N} \left(\sigma_i^2 \delta t_n - \Sigma_n^2\right) \Gamma^{(\sigma_i)}(t_{n-1}, S)$$

where $\Sigma_n = \left(\frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}\right)^2$ is the discrete realized variance

Recall that the standard deviation of a constant volatility parameter σ measured discretely is $\sigma/\sqrt{2N}$

The Expected P&L

Assume that hedging intervals are uniform with $\delta t_n \equiv \delta t = T/N$

The expected P&L:

$$\mathbb{M}_1(\{t_n\}) = \left(\sigma_i^2 - \sigma_a^2\right) \sum_{n=1}^{N} \delta t \Gamma^{(\sigma_i)}(t_{n-1}; S) \approx \left(\sigma_i^2 - \sigma_a^2\right) T \Gamma^{(\sigma_i)}(t_0, S)$$

In terms of option vega $\mathcal{V}(t_0, S)$, $\mathcal{V}(t_0, S) = 2\tilde{\sigma} T \Gamma^{(\sigma_i)}(t_0, S)$:

$$\mathbb{M}_1(\{t_n\}) \approx (\sigma_i - \sigma_a) \mathcal{V}(t_0, S)$$

The expected P&L equals to the spread between the implied and the actual volatilities multiplied by the option vega

The expected P&L does not depend on the hedging frequency!

The P&L volatility

Under the continuous hedging, the P&L volatility $\sigma_{\text{P\&L}}$ is affected only by the volatility of the asset price:

$$\sigma_{\text{P\&L}} \approx C\,|\sigma_i - \sigma_a|\,\mathcal{V}(t_0, S),$$

where C is a small number, $C \equiv C(\sigma_i, \sigma_a, t_0, S)$

In the ideal BSM model with $\sigma_i = \sigma_a$, $\sigma_{\text{P\&L}} = 0$

Under the discrete hedging, the P&L volatility $\sigma_{\text{P\&L}}$ is affected primarily by the volatility of the realized variance:

$$\sigma_{\text{P\&L}} \approx \sigma_a^2 \sqrt{\frac{1}{2N}} \frac{\mathcal{V}(t_0, S)}{\sigma_i}$$

For short-term at-the-money options:

$$\frac{\sigma_{\text{P\&L}}}{U^{(\sigma_i)}(t_0, S)} \approx \frac{\sigma_a^2}{\sigma_i^2} \sqrt{\frac{1}{2N}}$$

The P&L volatility is significant for short-term options relative to their value if N is small

Risks of the delta-hedging strategy

1) Mis-specification of volatility parameter (generally, mis-specification of the model dynamics)

2) Discrete hedging frequency

3) Transaction costs

4) Sudden jumps in the asset price

5) Volatility of the actual volatility σ_a (volatility of volatility)

For short-term options (with maturity up to one month), 1)-4) are significant

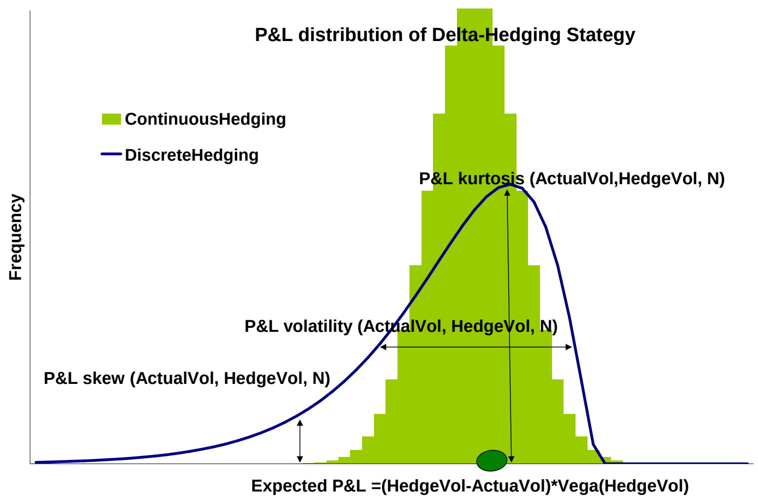
For longer-term options, 5) is important while 2) is less significant

In this talk, we concentrate on aspects 1)-4)

P&L distribution, Recall that the expected P&L does not depend on the hedging frequency: $\mathbb{M}_1(\{t_n\}) \approx (\sigma_i - \sigma_a)\,\mathcal{V}(t_0, S)$

We need to understand how the discrete hedging and jumps affect the higher moments of the P&L: volatility, skew, kurtosis

For this purpose we study the approximate distribution of the P&L rather than its higher moments (the latest is the common approach in the existing studies)



Path-dependence of the P&L

Under the discrete hedging, the P&L is a path-depended function:

$$P(T;\{t_n\}) = \sum_{n=1}^{N} \left(\sigma_i^2 \delta t_n - \Sigma_n^2\right) \Gamma^{(\sigma_i)}(t_{n-1}, S)$$

where $\Sigma_n = \left(\frac{S(t_n)-S(t_{n-1})}{S(t_{n-1})}\right)^2$ is the realized variance

Direct analytical methods are not applicable

What if we use the first order approximation in $\Gamma^{(\sigma_i)}(t_{n-1}, S)$?:

$$P(T;\{t_n\}) \approx \sum_{n=1}^{N} \left(\sigma_i^2 \delta t_n - \Sigma_n^2\right) \mathbb{G}^{(\sigma_i)}(t_{n-1}, S)$$

where $\mathbb{G}^{(\sigma_i)}(t_{n-1}, S) = \mathbb{E}[\Gamma^{(\sigma_i)}(t_{n-1}, S)]$

The approximate P&L is a function of the discrete realized variance

The first moment of the P&L is preserved, how is about its variance?

MC Simulations of the P&L

Apply Monte Carlo to simulate a path of $\{S_n\}$, $n = 1, \ldots N$, to compute

$$\text{P\&L}(T;N) = \sum_{n=1}^{N} \left[(S_n - S_{n-1})\Delta(t_{n-1}, S_{n-1}) - (U(t_n, S_n) - U(t_{n-1}, S_{n-1}))\right]$$

$$\text{RealizedVar}(T;N) = \sum_{n=1}^{N} \left(\frac{(S_n - S_{n-1})}{S_{n-1}}\right)^2$$

$$\text{RealizedGamma}(T;N) = \frac{T}{N}\sum_{n=1}^{N} \Gamma(t_{n-1}, S_{n-1})$$

$$\text{RealizedVarGamma}(T;N) = \sum_{n=1}^{N} \left(\frac{T\sigma_i^2}{N} - \left(\frac{(S_n - S_{n-1})}{S_{n-1}}\right)^2\right)\Gamma(t_{n-1}, S_{n-1})$$

"RealizedVarGamma" is an approximation to the P&L, how is about "RealizedVar"?

MC Simulations of the P&L

Call option with $S = K = 1$, $T = 0.08$ and $N = 5$ (weekly hedging), $N = 20$ (daily), $N = 80$ (every 1.5 hour), $N = 320$ (every 20 minutes)

The same model parameters are used to compute the delta-hedge and simulate $S(t)$ using three models:

1) BSM with $\sigma = 21.02\%$

2) Merton jump-diffusion (MJD) with $\sigma = 19.30\%$, $\lambda = 1.25$, $\nu = -7.33\%$, $\upsilon = (1.27\%)^2$

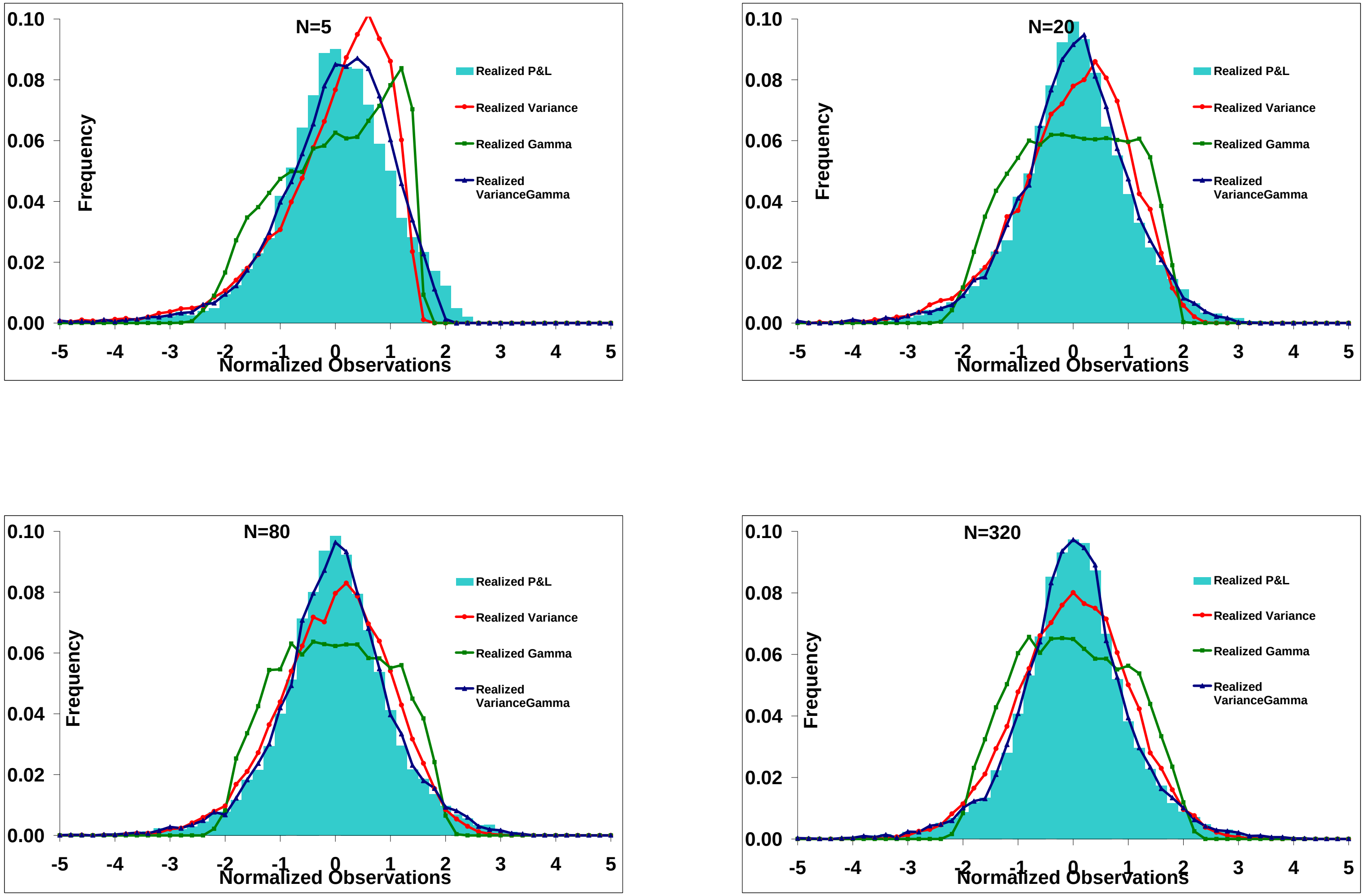
3) Heston SV with $V(0) = \theta = (21.02\%)^2$, $\kappa = 3$, $\varepsilon = 40\%$, $\rho = -0.7$

In MJD: $\sigma_i^2 = \sigma^2 + \lambda(\nu^2 + \upsilon)$; in Heston SV: $\sigma_i^2 = V(0)$

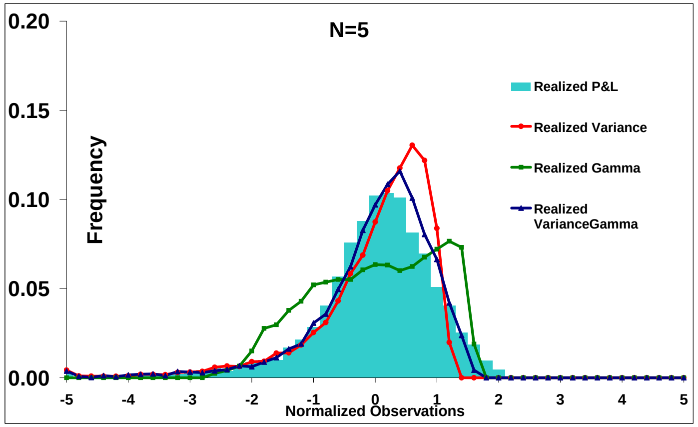
In MJD and Heston, option value, delta, and gamma are computed using efficient Fourier inversion methods by Lipton and Lewis

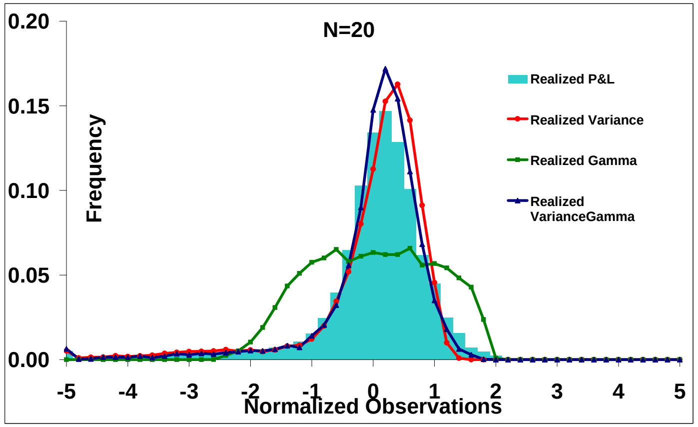
Simulated realizations are normalized to have zero mean and unit variance (with total of 10,000 paths)

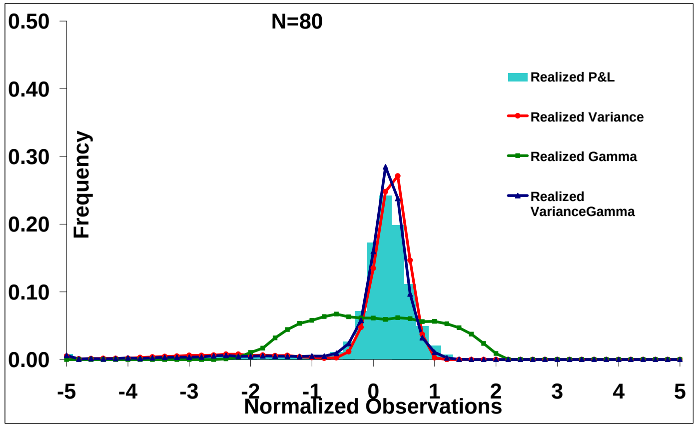
The P&L frequency in the BSM model

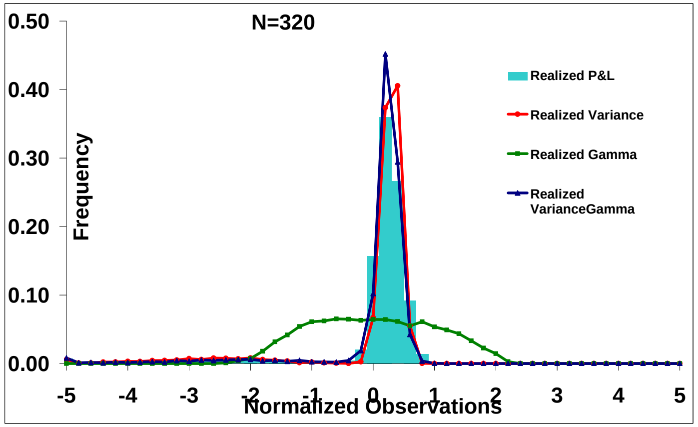


The P&L frequency in the Merton jump-diffusion model

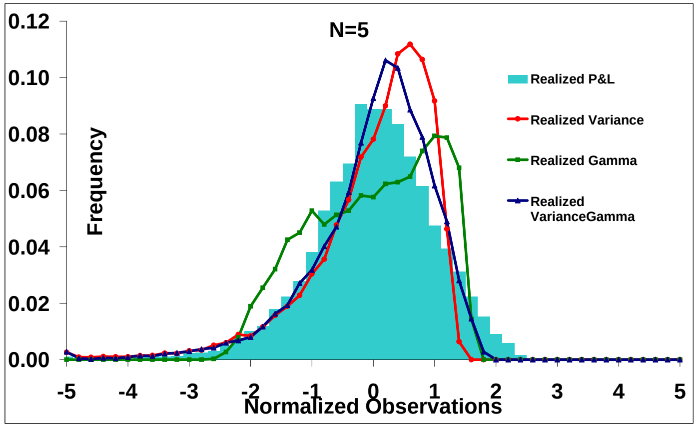


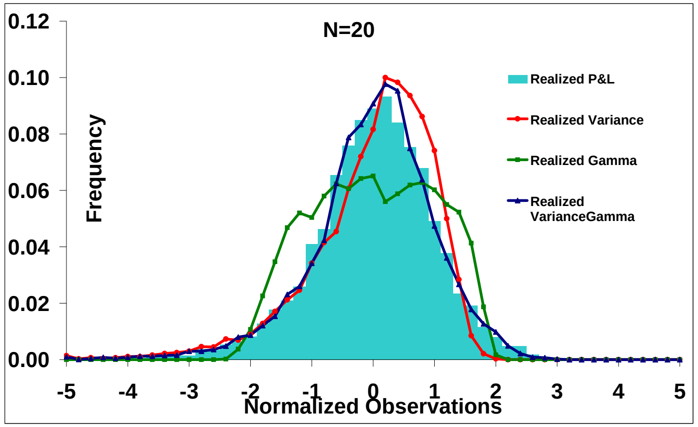


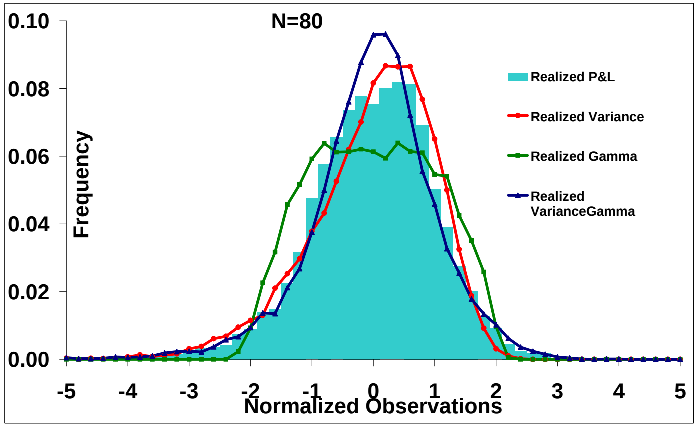


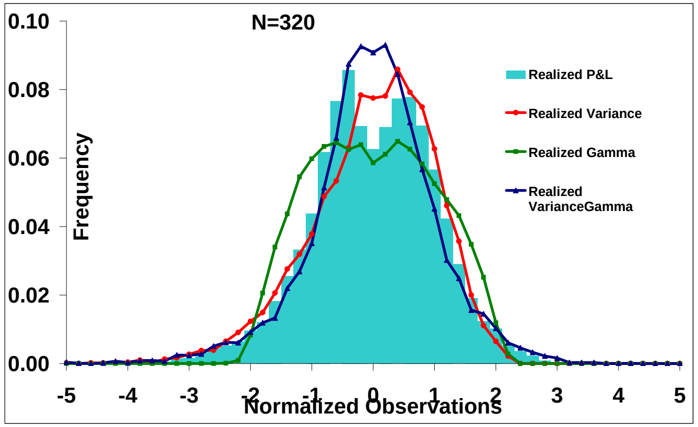


The P&L frequency in the Heston SV model









Observations

The realized variance gamma approximates the realized P&L closely if hedging is daily or more frequent

The frequency of realized variance scales with the frequency of the P&L

This is the key ground for our approximation: use the PDF of the realized variance to approximate the PDF of the realized P&L

The key tool: the Fourier transform of the PDF of the discrete realized variance

On side note: the P&L distribution in the BSM and Heston is close to the normal, while that in the MJD is peaked at zero with heavy left tail

Extended Merton Model. The JDM under the historic measure $\mathbb{P}$:

$$dS(t)/S(t) = \mu(t)dt + \sigma(t)dW(t) + \left(e^J - 1\right) dN(t), \ S(t_0) = S,$$

$N(t)$ is Poisson process with intensity $\lambda(t)$, jumps have the PDF $w(J)$:

$$w(J) = \sum_{l=1}^{L} p_l n(J; \nu_l, \upsilon_l), \ \sum_{l=1}^{L} p_l = 1,$$

L, $L = 1, 2, ..$, is the number of mixtures
p_l, $0 \leq p_l \leq 1$, is the probability of the l-th mixture
ν_l and υ_l are the mean and variance of the l-th mixture, $l = 1, .., L$

Under the pricing measure $\mathbb{Q}$, the JDM for $S(t)$ has volatility $\widetilde{\sigma}(t)$, jump intensity $\widetilde{\lambda}(t)$, risk-neutral drift $\widetilde{\mu}(t)$:

$$\widetilde{\mu}(t) = r(t) - d(t) - \widetilde{\lambda}(t)\mathbb{J}^{\mathbb{Q}},$$

$$\widetilde{w}(J) = \sum_{l=1}^{L} \widetilde{p}_l n(J; \widetilde{\nu}_l, \widetilde{\upsilon}_l), \ \sum_{l=1}^{L} \widetilde{p}_l = 1,$$

$$\mathbb{J}^{\mathbb{Q}} \equiv \int_{-\infty}^{\infty} \left(e^J - 1\right) \widetilde{w}(J) dJ = \sum_{l=1}^{L} \widetilde{p}_l e^{\widetilde{\nu}_l + \frac{1}{2}\widetilde{\upsilon}_l} - 1$$

Notations

Define the time-integrated quantities, with $0 \leq t_0 \leq t$

The variance under $\mathbb{P}$ and $\mathbb{Q}$, respectively:

$$\overline{\vartheta}(t;t_0) = \int_{t_0}^{t} \sigma^2(t')dt',\ \overline{\widetilde{\vartheta}}(t;t_0) = \int_{t_0}^{t} \widetilde{\sigma}^2(t')dt'$$

The risk-free rate and dividend yield:

$$\overline{r}(t;t_0) = \int_{t_0}^{t} r(t')dt',\ \overline{d}(t;t_0) = \int_{t_0}^{t} d(t')dt'$$

The drift under $\mathbb{P}$ and $\mathbb{Q}$, respectively:

$$\overline{\mu}(t;t_0) = \int_{t_0}^{t} \mu(t')dt',\ \overline{\widetilde{\mu}}(t;t_0) = \int_{t_0}^{t} \widetilde{\mu}(t')dt'$$

The intensity rate under $\mathbb{P}$ and $\mathbb{Q}$, respectively:

$$\overline{\lambda}(t;t_0) = \int_{t_0}^{t} \lambda(t')dt',\ \overline{\widetilde{\lambda}}(t;t_0) = \int_{t_0}^{t} \lambda(t')dt'$$

Transition PDF of x(t), $x(t) = \ln S(t)$ and $X = x(T)$:

$$G^{\mathsf{X}}(T, X; t, x) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_L=0}^{\infty} \mathcal{P}(m_1; p_1\overline{\lambda}(T;t))\ldots\mathcal{P}(m_L; p_L\overline{\lambda}(T;t)) n(X; \alpha, \beta),$$

$$\alpha \equiv \alpha(m_1, ..., m_L) = x + \overline{\mu}(T;t) - \frac{1}{2}\overline{\vartheta}(T;t) + \sum_{l=1}^{L} m_l \nu_l,$$

$$\beta \equiv \beta(m_1, ..., m_L) = \overline{\vartheta}(T;t) + \sum_{l=1}^{L} m_l \upsilon_l,$$

where $n(x; \alpha, \beta)$ is the PDF of a normal with mean α and variance β
$\mathcal{P}(m; \lambda)$, $\mathcal{P}(m; \lambda) = e^{-\lambda}\lambda^m/m!$, is the Poisson probability
Call options are valued by the extension of Merton formula (1976):

$$U(t, S; T, K) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_L=0}^{\infty} \mathcal{P}(m_1; p_1\overline{\overline{\lambda}}(T;t)(\mathbb{J}^{\mathbb{Q}} + 1))\ldots\mathcal{P}(m_L; p_L\overline{\overline{\lambda}}(T;t)(\mathbb{J}^{\mathbb{Q}} + 1$$

$$\times\, \mathcal{C}^{(BSM)}(S, K, 1; \widetilde{\beta}(m_1, ..., m_L), \widetilde{\alpha}(m_1, ..., m_L), \overline{d}(T;t)),$$

$$\widetilde{\alpha}(m_1, ..., m_L) = \overline{r}(T;t) - \overline{\overline{\lambda}}(T;t)\mathbb{J}^{\mathbb{Q}} + \sum_{l=1}^{L} m_l \widetilde{\nu}_l + \frac{1}{2}\sum_{l=1}^{L} m_l \widetilde{\upsilon}_l, \;\; \widetilde{\beta}(m_1, ..., m_L) = \overline{\overline{\vartheta}}($$

Empirical Estimation

Use closing levels of the S&P500 index from January 4, 1999, to January 9, 2009, with the total of 2520 observations

Fit empirical quantiles of sampled daily log-returns to model quantiles by minimizing the sum of absolute differences between the two

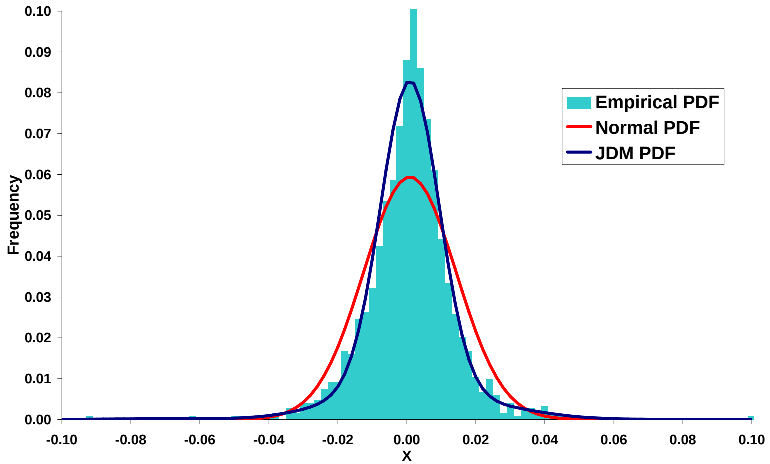
Assume time-homogeneous parameters, $\mu = 0$, and jump variance v_l uniform across different states

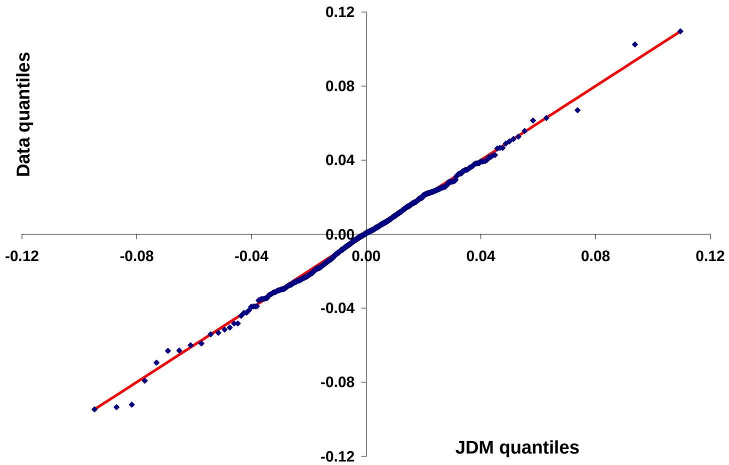
The best fit, keeping L as small as possible, is obtained with $L = 4$

Empirical Estimation

Model estimates: $\sigma = 0.1348$, $\lambda = 46.4444$

l	p_l	ν_l	$\sqrt{\upsilon_l}$	$p_l\lambda$	expected frequency
1	0.0208	-0.0733	0.0127	0.9641	every year
2	0.5800	-0.0122	0.0127	26.9399	every two weeks
3	0.3954	0.0203	0.0127	18.3661	every six weeks
4	0.0038	0.1001	0.0127	0.1743	every five years





Assumptions for the delta-hedging strategy

1) The synthetic market is based on the continuous-time model of the asset price dynamics specified by either the diffusion model (DM) or the jump-diffusion model (JDM) with known model parameters under $\mathbb{P}$ and $\mathbb{Q}$

2) The option value and delta are computed using model parameters under $\mathbb{Q}$; the PDF of the P&L is computed using model parameters under $\mathbb{P}$

3) The hedging strategy is time based with the set of trading times $\{t_n\}$, trading intervals (not necessarily uniform) defined by $\{\delta t_n\}$, where $\delta t_n = t_n - t_{n-1}$, $n = 1, ..., N$, and N is the total number of trades, $t_0 = 0$, $t_N = T$

Given the discrete time grid $\{t_n\}$, we define: $\mu_n = \overline{\mu}(t_n; t_{n-1})$, $\vartheta_n = \overline{\vartheta}(t_n; t_{n-1})$, $\widetilde{\vartheta}_n = \overline{\overline{\vartheta}}(t_n; t_{n-1})$, $\lambda_n = \overline{\lambda}(t_n; t_{n-1})$, $\widetilde{\lambda}_n = \overline{\overline{\lambda}}(t_n; t_{n-1})$

Discrete model

Discrete version of the DM under $\mathbb{P}$ on the grid $\{t_n\}$:

$$S(t_n) = S(t_{n-1}) \exp\left\{\mu_n - \frac{1}{2}\vartheta_n + \sqrt{\vartheta_n}\epsilon_n\right\},$$

where $\{\epsilon_n\}$ is a collection of independent standard NRVs, $n = 1, ..., N$

For small μ_n and $\sqrt{\vartheta_n}$ are small, we approximate:

$$S(t_n) \approx S(t_{n-1})\left(1 + \mu_n + \sqrt{\vartheta_n}\epsilon_n\right)$$

Hedging portfolio $\Pi(t_n, S)$ at rebalancing time t_n:

$$\Pi(t_n, S) = S(t_n)\Delta(t_n, S) - U(t_n, S)$$

The P&L at time t_n, $\delta\Pi_n$:

$$\delta\Pi_n \approx \frac{1}{2}\left(\widetilde{\vartheta}_n - \Sigma_n^2\right) S^2(t_{n-1}) U_{SS}(t_{n-1}, S),$$

$$\Sigma_n^2 \equiv \frac{(S(t_n) - S(t_{n-1}))^2}{S^2(t_{n-1})} \approx \left(\mu_n + \sqrt{\vartheta_n}\epsilon_n\right)^2$$

Transaction costs

Proportional costs is the bid-ask spread:

$$\kappa = 2\frac{S_{\text{ask}} - S_{\text{bid}}}{S_{\text{ask}} + S_{\text{bid}}},$$

where S_{ask} and S_{bid} are the quoted ask and bid prices, respectively

Thus, $\kappa/2$ is the average percentage loss per trade amount and:

$$S_{\text{ask}}(t) = (1 + \kappa/2)S(t),\ S_{\text{bid}}(t) = (1 - \kappa/2)S(t)$$

Θ_n is transaction costs due to rebalancing of delta-hedge:

$$\Theta_n \equiv (\kappa/2)S(t_n)\,|\Delta(t_n, S) - \Delta(t_{n-1}, S)|$$

Approximation for the transaction costs

Apply Taylor expansion for $U_S(t_n, S)$:

$$\Theta_n \approx (\kappa/2) S(t_n) U_{SS}(t_{n-1}, S) \left|\delta S_n\right|$$

Apply approximation for δS_n, we obtain:

$$\Theta_n \approx (\kappa/2) S^2(t_{n-1}) U_{SS}(t_{n-1}, S) \left| \left(\frac{1}{2} + \mu_n + \sqrt{\vartheta_n} \epsilon_n \right)^2 - \frac{1}{4} \right|$$

Define the following function:

$$F(\alpha, \beta) \equiv \mathbb{E}_\epsilon \left[\left| \left(\frac{1}{2} + \alpha + \sqrt{\beta} \epsilon \right)^2 - \frac{1}{4} \right| \right] = \sqrt{\frac{2\beta}{\pi}} \left((\alpha + 1) e^{-\frac{\alpha^2}{2\beta}} - \alpha e^{-\frac{(1+\alpha)^2}{2\beta}} \right.$$
$$\left. + 2(\beta + \alpha^2 + \alpha) \left(\frac{1}{2} + \mathcal{N}\left(\frac{\alpha}{\sqrt{\beta}} \right) - \mathcal{N}\left(\frac{1+\alpha}{\sqrt{\beta}} \right) \right) \right)$$

Approximate Θ_n by (the first-order moment matching):

$$\Theta_n \approx \frac{1}{2} \theta_n S^2(t_{n-1}) U_{SS}(t_{n-1}, S),$$

where $\theta_n = \overline{\kappa}_n \epsilon_n^2$, $\overline{\kappa}_n \equiv \kappa F\left(\mu_n, \vartheta_n\right)$

Approximation for the total P&L

The total P&L, $P(\{t_n\})$, is obtained by accumulating $\Delta\Pi_n$:

$$P(\{t_n\}) \equiv \sum_{n=1}^{N} (\delta\Pi_n - \Theta_n)$$

Apply our approximations:

$$P(\{t_n\}) \approx \sum_{n=1}^{N} \left(\tilde{\vartheta}_n - \Sigma_n^2 - \theta_n\right) \Gamma(t_{n-1}, S),$$

where $\Gamma(t, S)$ is the option cash-gamma: $\Gamma(t, S) = \frac{1}{2} S^2(t) U_{SS}(t, S)$

Recall that $\Sigma_n^2 \approx \left(\mu_n + \sqrt{\vartheta_n}\epsilon_n\right)^2$

Explicitly, the P&L is a quadratic function of $\{\epsilon_n\}$, $n = 1, ..., N$:

$$P(\{t_n\}) = \sum_{n=1}^{N} M(\epsilon_n; A_n, B_n, C_n, -\Gamma(t_{n-1}, S)),$$

$M(\epsilon; A, B, C, o) \equiv o\left(A + C\epsilon + B\epsilon^2\right)$

$A_n = -\tilde{\vartheta}_n + \mu_n^2,\ C_n = 2\mu_n\sqrt{\vartheta_n},\ B_n = (\vartheta_n + \overline{\kappa}_n)$

Expected Cash-Gamma

For vanilla calls and puts with strike K:

$$\Gamma(t,S) = \frac{K}{2\sqrt{2\pi\overline{\widetilde{\vartheta}}(T;t)}} \exp\left\{ -\frac{1}{2\overline{\widetilde{\vartheta}}(T;t)} \left(\ln\frac{S}{K} - \frac{1}{2}\overline{\widetilde{\vartheta}}(T;t) \right)^2 \right\}$$

Under $\mathbb{P}$:

$$S(t_n) = S(t_0)\exp\left\{ \overline{\mu}(t_n;t_0) - \frac{1}{2}\overline{\vartheta}(t_n;t_0) + \sqrt{\overline{\vartheta}(t_n;t_0)}\varepsilon_n \right\}$$

where ε_n is a standard NRV

The expected cash gamma under $\mathbb{P}$, $\mathbb{G}(T;t)$:

$$\mathbb{G}(t_n;t_0) \equiv \mathbb{E}^{\mathbb{P}}\left[\Gamma(t_n,S)\right] = \mathbb{E}_{\varepsilon_n}\left[\Gamma(t_n,S) \,|\, S = S(t_n)\right]$$
$$= \frac{K}{2\sqrt{2\pi\zeta(t_0,t_n,T)}} \exp\left\{ -\frac{1}{2\zeta(t_0,t_n,T)} \left(\ln\frac{S}{K} - \frac{1}{2}\zeta(t_0,t_n,T) \right)^2 \right\},$$

where $\zeta(t_0,t_n,T) = \overline{\vartheta}(t_n;t_0) + \overline{\widetilde{\vartheta}}(T;t_n)$

If model parameters under $\mathbb{Q}$ and $\mathbb{P}$ are equal: $\mathbb{G}(t_n;t_0) = \Gamma(t_0,S)$

The Fourier transform of the quadratic function:

$$Z(A,B,C;o) \equiv \mathbb{E}_\epsilon\left[e^{-M(\epsilon;A,B,C,o)}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-oA - oCx - oBx^2 - \frac{x^2}{2}\right\} dx$$

$$= \frac{1}{\sqrt{2oB+1}} \exp\left\{\frac{\frac{1}{2}(oC)^2}{2oB+1} - oA\right\}$$

Consider the following sum:

$$I(N) = \sum_{n=1}^{N} M(\epsilon_n; A_n, B_n, C_n, o_n)$$

where $\{\epsilon_n\}$ is a set of independent standard NRVs, A_n, B_n, C_n are reals and o_n is complex with $\Re[o_n B_n] > -1/2$, $n = 1, ..., N$
Compute the Fourier transform of $I(N)$, G^{I}:

$$G^{\mathrm{I}} \equiv \mathbb{E}_{\{\epsilon_n\}}\left[e^{-I(N)}\right] = \prod_{n=1}^{N} \mathbb{E}_{\epsilon_n}\left[e^{-M(\epsilon_n;A_n,B_n,C_n,o_n)}\right] = \prod_{n=1}^{N} Z(A_n, B_n, C_n; o_n)$$

$$= \exp\left\{\sum_{n=1}^{N}\left(\frac{\frac{1}{2}(o_n C)_n^2}{2o_n B_n + 1} - o_n A_n\right)\right\} \prod_{n=1}^{N} \frac{1}{\sqrt{2o_n B_n + 1}}$$

The Fourier transform of the P&L distribution

Denote the PDF of the P&L $P(\{t_n\})$ by $G^P(P'; \{t_n\})$

Denote its Fourier transform under $\mathbb{P}$ by $\widehat{G}^P(\Psi; \{t_n\})$ with $\Psi = \Psi_R + i\Psi_I$, where Ψ_R and Ψ_I are reals and $i = \sqrt{-1}$:

$$\begin{aligned}\widehat{G}^P(\Psi; \{t_n\}) &\equiv \mathbb{E}^{\mathbb{P}}\left[e^{-\Psi P(\{t_n\})}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left\{-\Psi \sum_{n=1}^{N} M(\epsilon_n; A_n, B_n, C_n; -1)\Gamma(t_{n-1}, S)\right\}\right]\end{aligned}$$

Approximate $\widehat{G}^P$ by:

$$\begin{aligned}\widehat{G}^P(\Psi; \{t_n\}) &\approx \mathbb{E}^{\mathbb{P}}\left[\exp\left\{-\Psi \sum_{n=1}^{N} M(\epsilon_n; A_n, B_n, C_n, -1)\mathbb{E}^{\mathbb{P}}\left[\Gamma(t_{n-1}, S)\right]\right\}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left\{-\sum_{n=1}^{N} M(\epsilon_n; A_n, B_n, C_n, \Psi_n^*)\right\}\right] \\ &= \prod_{n=1}^{N} Z(A_n, B_n, C_n; \Psi_n^*),\end{aligned}$$

where $\Psi_n^* = -\Psi\mathbb{G}(t_{n-1}; t_0)$

The Fourier transform of the P&L distribution

Simplify:

$$\widehat{G}^P(\Psi;\{t_n\}) \approx \exp\left\{\sum_{n=1}^{N}\left(-\frac{\Psi_n^*\mu_n^2}{2\Psi_n^*(\vartheta_n+\overline{\kappa}_n)+1}+\widetilde{\vartheta}_n\Psi_n^*\right)\right\}$$
$$\times\prod_{n=1}^{N}\frac{1}{\sqrt{2\Psi_n^*(\vartheta_n+\overline{\kappa}_n)+1}}$$

This expression can be recognized as the characteristic function of non-central chi-square distribution with in-homogeneous drift, variance, and shift parameters.

The jump-diffusion model. The Fourier transform of the approximate P&L distribution is (for derivation, see my paper):

$$\widehat{G}^P(\Psi;\{t_n\}) \equiv \mathbb{E}^{\mathbb{P}}\left[e^{-\Psi P(\{t_n\})}\right]$$

$$\approx \prod_{n=1}^{N} \sum_{m_1=0}^{\infty} ... \sum_{m_L=0}^{\infty} \mathcal{P}(m_1; p_1\lambda_n)...\mathcal{P}(m_L; p_L\lambda_n) Z(A_n - \widetilde{\Sigma}_n^2, B_n, C_n; -\mathbb{G}(t_{n-1}; t_0)$$

$$A_n \equiv A_n(m_1, ..., m_L) = \alpha_n^2(m_1, ..., m_L),$$

$$B_n \equiv B_n(m_1, ..., m_L) = \beta_n(m_1, ..., m_L),$$

$$C_n \equiv C_n(m_1, ..., m_L) = \alpha_n(m_1, ..., m_L)\sqrt{\beta_n(m_1, ..., m_L)},$$

$$\alpha_n(m_1, ..., m_L) = \mu_n + \overline{\nu}(m_1, ..., m_L),$$

$$\beta_n(m_1, ..., m_L) = \vartheta_n + \overline{\upsilon}(m_1, ..., m_L) + \overline{\kappa}_n(m_1, ..., m_L),$$

$$\overline{\kappa}_n(m_1, ..., m_L) \equiv \kappa F\left(\mu_n + \overline{\nu}(m_1, ..., m_L), \vartheta_n + \overline{\upsilon}(m_1, ..., m_L)\right),$$

$$\overline{\nu}(m_1, ..., m_L) \equiv \sum_{l=1}^{L} m_l \nu_l,\ \ \overline{\upsilon}(m_1, ..., m_L) \equiv \sum_{l=1}^{L} m_l \upsilon_l,$$

$$\widetilde{\Sigma}_n^2 = \widetilde{\vartheta}_n + \widetilde{\lambda}_n \sum_{l=1}^{L} \widetilde{p}_l \left(e^{2\widetilde{\nu}_l + 2\widetilde{\upsilon}_l} - 2e^{\widetilde{\nu}_l + \frac{1}{2}\widetilde{\upsilon}_l} + 1\right)$$

Implications 0

Using the Fourier transform of the approximate distribution of the P&L, we can compute the approximate PDF of P:

$$G^P(P'; \{t_n\}) = \frac{1}{\pi} \int_0^{\infty} \Re \left[e^{\Psi P'} \widehat{G}^P(\Psi) \right] d\Psi_I,$$

where we apply standard FFT methods to compute the inverse Fourier transform

Also, we can compute the m-th moment of P by:

$$\mathbb{E}^{\mathbb{P}} \left[(P(\{t_n\}))^m \right] = (-1)^m \left. \frac{\partial^m \widehat{G}^P(\Psi; \{t_n\})}{\partial \Psi_R^m} \right|_{\Psi_R = 0, \Psi_I = 0}, \ m = 1, 2, ...$$

Implications I

The expected P&L:

$$\mathbb{M}_1(\{t_n\}) = \sum_{n=1}^{N} \left(\widetilde{\vartheta}_n - \vartheta_n - \overline{\kappa}_n - \mu_n^2\right) \mathbb{G}(t_{n-1}; t_0)$$

If the hedging frequency is uniform with $\delta t = T/N$ and model parameters are constant:

$$\mathbb{M}_1(\{t_n\}) \approx \left(\widetilde{\sigma}^2 - \sigma^2 - \kappa\sigma\sqrt{\frac{2N}{\pi T}} - \frac{\mu^2 T}{N}\right) T\Gamma(t_0, S)$$

The break-even volatility $\widetilde{\sigma}$, so that $\mathbb{M}_1(\{t_n\}) = 0$:

$$\widetilde{\sigma}^2 = \sigma^2 + \kappa\sigma\sqrt{\frac{2N}{\pi T}},$$

which is Leland (1985) volatility adjustment for a short option position

Implications II

Assume $\delta t \equiv \delta t_n = T/N$ and parameters under $\mathbb{Q}$ and $\mathbb{P}$ are equal with:

$$\mu_n = \int_{t_{n-1}}^{t_n} \mu dt \equiv \mu\delta t,\ \ \vartheta_n = \widetilde{\vartheta}_n = \int_{t_{n-1}}^{t_n} \sigma^2 dt \equiv \sigma^2 \delta t$$

The Fourier transform of the P&L distribution

$$\widehat{G}^P(\Psi; \{t_n\}) = \frac{1}{(2\Psi^\circ + 1)^{N/2}} \exp\left(-\frac{\eta\Psi^\circ}{2\Psi^\circ + 1} + s\Psi^\circ\right)$$

$$\Psi^\circ = -c\Gamma(t_0, S)\Psi,\ \ c = \frac{\sigma^2 T}{N} + \kappa\sqrt{\frac{2}{\pi}\frac{\sigma^2 T}{N}},\ \ \eta = \frac{\mu^2 T^2}{Nc},\ \ s = \frac{\sigma^2 T}{c}.$$

This is the Fourier transform of the non-central chi-square distribution with N degrees of freedom, non-centrality parameter η, $\chi(y, N, \eta)$:

$$G^P(P'; \{t_n\}) = \frac{1}{\widetilde{c}}\chi\left(\frac{\widetilde{s} - P'}{\widetilde{c}}, N, \eta\right) \mathbf{1}_{\{P' \le \widetilde{s}\}}$$

where $\widetilde{s} = \sigma^2 T\Gamma(t_0, S)$, $\widetilde{c} = c\Gamma(t_0, S)$

Implications III

The P&L $\mathbb{M}_1$:

$$\mathbb{M}_1 = -\Gamma(t_0, S)\left(\frac{\mu^2 T^2}{N} + \kappa\sqrt{\frac{2\sigma^2 TN}{\pi}}\right)$$

The P&L volatility $\mathbb{M}_2$ is inversely proportional to $\sqrt{N}$:

$$(\mathbb{M}_2)^2 = \Gamma^2(t_0, S)(T\sigma^2)\left(\frac{2T\sigma^2}{N} + \frac{4\kappa\sqrt{2T\sigma^2}}{\sqrt{\pi N}} + \frac{4\kappa^2}{\pi} + O\left(\frac{1}{N^{3/2}}\right)\right)$$

The skewness $\mathbb{M}_3$ and the excess kurtosis $\mathbb{M}_4$:

$$\mathbb{M}_3 = -\frac{2^{3/2}}{\sqrt{N}} + O\left(\frac{1}{N}\right),\ \mathbb{M}_4 = \frac{12}{N} + O\left(\frac{1}{N^{2/3}}\right)$$

As $N \to \infty$, the chi-squared distribution converges to normal:

$$G^P(P'; \{t_n\}) \approx n(P'; \mathbb{M}_1, (\mathbb{M}_2)^2)$$

Jump-diffusion model I
The expected P&L, $\mathbb{M}_1(\{t_n\})$:

$$\mathbb{M}_1(\{t_n\}) = \sum_{n=1}^{N} \left(\widetilde{\vartheta}_n - \vartheta_n + \widetilde{\lambda}_n \mathbb{E}^{\mathbb{Q}}[J^2] - \lambda_n \mathbb{E}^{\mathbb{P}}[J^2] - \mu_n^2 - \mathbb{K}_n\right) \mathbb{G}(t_{n-1}; t_0)$$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[J^2] &\equiv \int_{-\infty}^{\infty} \left(e^J - 1\right)^2 \widetilde{w}(J) dJ \\ &= \sum_{l=1}^{L} \widetilde{p}_l \left(e^{2\widetilde{\nu}_l + 2\widetilde{\upsilon}_l} - 2e^{\widetilde{\nu}_l + \frac{1}{2}\widetilde{\upsilon}_l} + 1\right) \approx \sum_{l=1}^{L} \widetilde{p}_l \left(\widetilde{\nu}_l^2 + \widetilde{\upsilon}_l\right)\end{aligned}$$

$\mathbb{E}^{\mathbb{P}}[J^2]$ computed under $\mathbb{P}$ by analogy

$\mathbb{K}_n$ is the expected proportional transaction costs

$$\begin{aligned}\mathbb{K}_n \equiv \kappa & \sum_{m_1=0}^{\infty} ... \sum_{m_L=0}^{\infty} \mathcal{P}(m_1; p_1\lambda_n)...\mathcal{P}(m_L; p_L\lambda_n) \\ & \times F(\mu_n + \overline{\nu}(m_1, ..., m_L), \vartheta_n + \overline{\upsilon}(m_1, ..., m_L))\end{aligned}$$

The hedger should match $\widetilde{\vartheta}_n$ and ϑ_n, $\mathbb{E}^{\mathbb{Q}}[J^2]$ and $\mathbb{E}^{\mathbb{P}}[J^2]$, $\widetilde{\lambda}_n$ and λ_n

Jump-diffusion model II

Assume constant model parameters equal under $\mathbb{Q}$ and $\mathbb{P}$, $\delta t = T/N$, a single jump component, $L = 1$, with mean and variance ν and υ

Approximate the PDF of the Poisson process by Bernoulli distribution taking $\mathcal{P}(0;\lambda_n) = 1 - \lambda_n$, $\mathcal{P}(1;\lambda_n) = \lambda_n$, and $\mathcal{P}(m;\lambda_n) = 0$ for $m > 1$

The expected total transaction costs $\mathbb{K}(N)$:

$$\mathbb{K}(N) \equiv \Gamma(t_0, S) \sum_{n=1}^{N} \mathbb{K}_n \approx \kappa\Gamma(t_0, S) \left(\sqrt{N}\sqrt{\frac{2\sigma^2 T}{\pi}} + \lambda T \sqrt{\frac{2(\nu^2 + \upsilon)}{\pi}} \right)$$

The first term is the expected transaction costs due to the diffusion

The second term represents those due to jumps

Jump-diffusion model III

If the hedging frequency is uniform with $\delta t = T/N$ and model parameters are constant:

$$\mathbb{M}_1(\{t_n\}) \approx \left(\widetilde{\sigma}^2 - \sigma^2 + \widetilde{\lambda}(\widetilde{\nu}^2 + \upsilon) - \lambda(\nu^2 + \widetilde{\upsilon}) - \kappa\left(\sigma\sqrt{\frac{2N}{\pi T}} - \lambda\sqrt{\frac{2(\nu^2+\upsilon)}{\pi}}\right)\right) \times T\Gamma(t_0, S)$$

The break-even volatility $\widetilde{\sigma}$, so that the expected P&L is zero:

$$\widetilde{\sigma}^2 = \sigma^2 + \lambda(\nu^2 + \upsilon) - \widetilde{\lambda}(\widetilde{\nu}^2 + \widetilde{\upsilon}) + \kappa\sigma\sqrt{\frac{2N}{\pi T}} + \kappa\lambda\sqrt{\frac{2(\nu^2+\upsilon)}{\pi}},$$

which is can be interpreted as the Leland volatility adjustment for the Merton JDM

Jump-diffusion model IV

The expected P&L without transaction costs:

$$\mathbb{M}_1 = -\Gamma(t_0, S)\left(\frac{\mu^2 T^2}{N} + \frac{2\lambda\mu\nu T^2}{N}\right)$$

The P&L volatility:

$$(\mathbb{M}_2)^2 = \Gamma^2(t_0, S)\left(\lambda T\left(\nu^4 + 3\upsilon\nu^2 + 3\upsilon^2\right) + \frac{\widehat{c}}{N} + O\left(\frac{1}{N^2}\right)\right)$$

where $\widehat{c} = \left(2\sigma^4 + \lambda\sigma^2(\nu^2 + \upsilon) - \lambda^2(\nu^2 + \upsilon)^2 + \lambda\mu(6\nu\upsilon + 4\nu^3)\right)T^2$

When jumps are present the volatility of the P&L cannot be reduced by increasing the hedging frequency

The constant term is proportional to expected number of jumps λT

The sign of the jump is irrelevant

The skewness $\mathbb{M}_3$ and the excess kurtosis $\mathbb{M}_4$:

$$\lim_{N\to\infty} \mathbb{M}_3 = -O\left(\frac{1}{\sqrt{\lambda T}}\right), \ \lim_{N\to\infty} \mathbb{M}_4 = O\left(\frac{1}{\lambda T}\right)$$

Mean-Variance Analysis I

The hedger's objective is to minimize utility function:

$$\min_N \left\{ \mathbb{K}(N) + \frac{1}{2\gamma}\mathbb{V}(N) \right\}$$

where γ is the risk-aversion tolerance, $\mathbb{K}(N)$ is the expected total hedging costs and $\mathbb{V}(N)$ is the variance of the P&L

We have shown that for both DM and JDM models:

$$\mathbb{K}(N) = k_0 + k_1\sqrt{N},\ \mathbb{V}(N) = v_0 + \frac{v_1}{N}$$

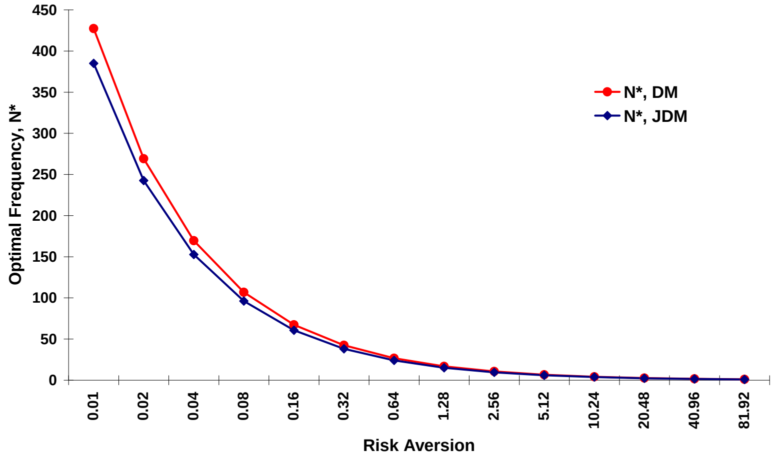
Thus, the optimal solution N^* is given by:

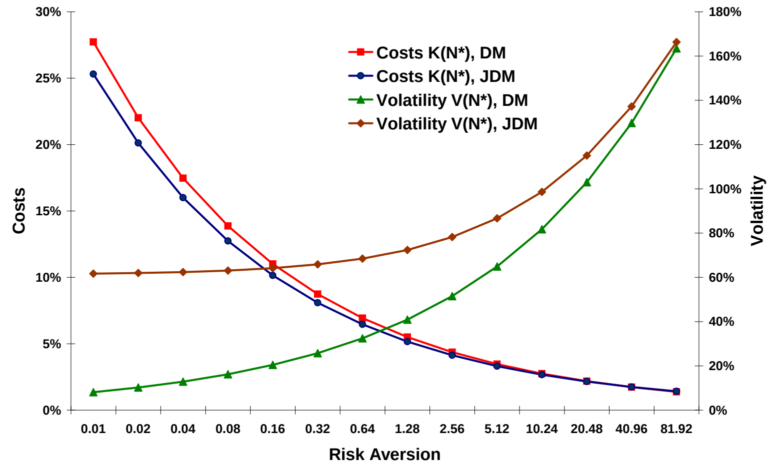
$$N^* = \left(\frac{v_1}{\gamma k_1}\right)^{2/3}$$

For the DM: $N^* = (2\pi)^{1/3}\sigma^2 T \left(\frac{\Gamma(t,S)}{\gamma\kappa}\right)^{2/3}$,

so that N^* is proportional to variance $\sigma^2 T$ and $\Gamma(t_0, S)$, and inversely proportional to the risk-aversion tolerance and transaction costs

Mean-Variance Analysis II





Left side: the optimal hedging frequency N^*, Right side: the expected transaction costs $\mathbb{K}(N^*)$ normalized by the option value as function of the risk-aversion tolerance and the P&L volatility $\sqrt{\mathbb{V}(N^*)}$ under the DM (DM) and JDM (JDM) using $\kappa = 0.002$ for a put option with $S(0) = K = 1.0$ and maturity $T = 0.08$

For a small level of the risk tolerance, the expected transaction costs amount to about $20 - 30\%$ of the option premium

P&L Distribution. Illustrations

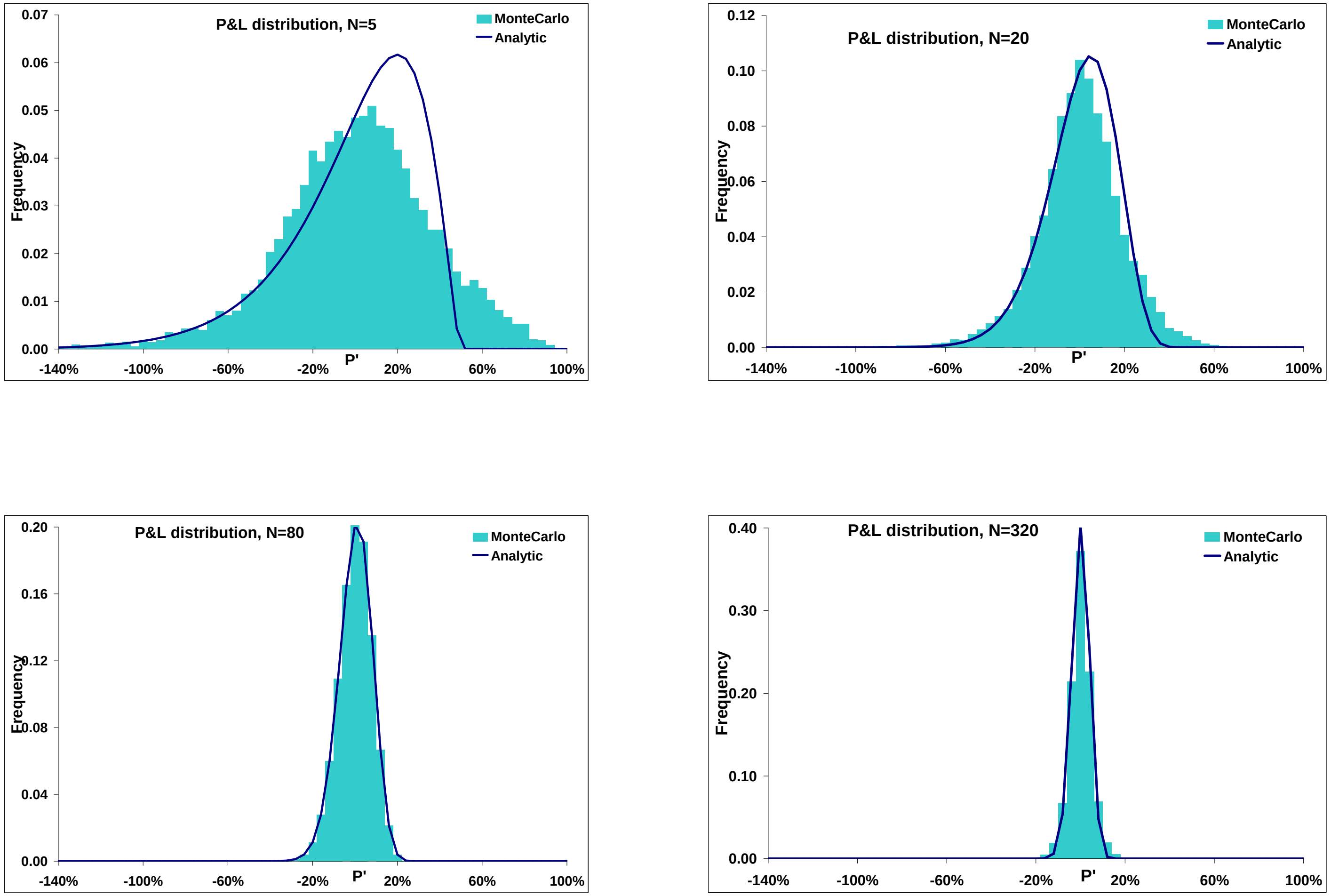
Use two models:
1) BSM with $\sigma = 21.02\%$
2) Merton jump-diffusion (MJD) with $\sigma = 19.30\%$, $\lambda = 1.25$, $\nu = -7.33\%$, $\upsilon = (1.27\%)^2$

Apply two cases:
1) transaction costs are zero
2) transaction costs with rate $\kappa = 0.002$

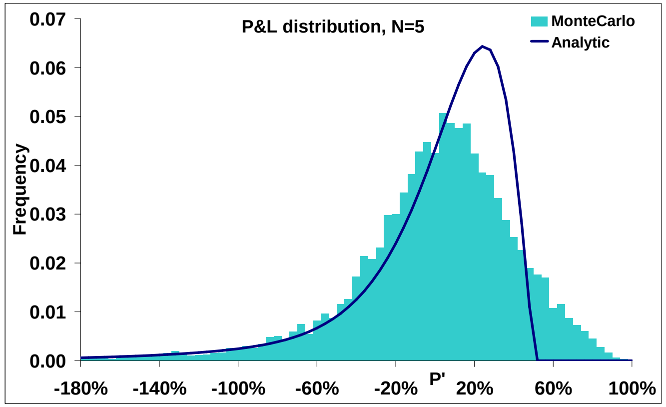
Call option with $S = K = 1$, $T = 0.08$ and $N = 5$ (weekly hedging), $N = 20$ (daily), $N = 80$ (every 1.5 hour), $N = 320$ (every 20 minutes)

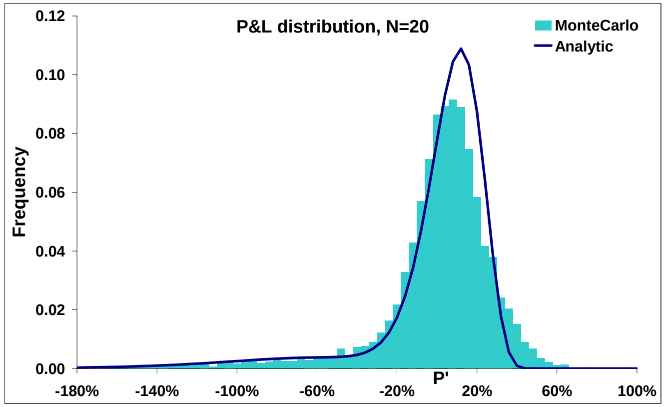
The P&L distribution is obtained using the Fourier inversion of the analytic approximations (Analytic) and Monte Carlo simulations (MonteCarlo) with total of 10,000 paths

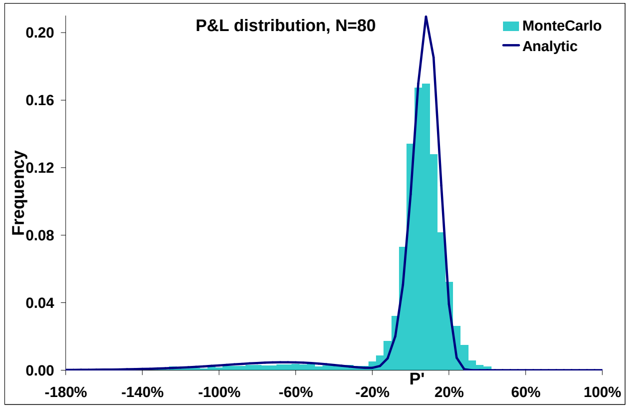
P&L Distribution in the DM without transaction costs

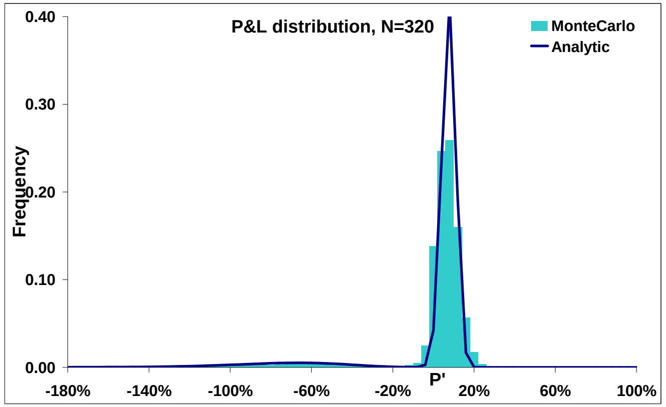


P&L Distribution in the JDM without transaction costs

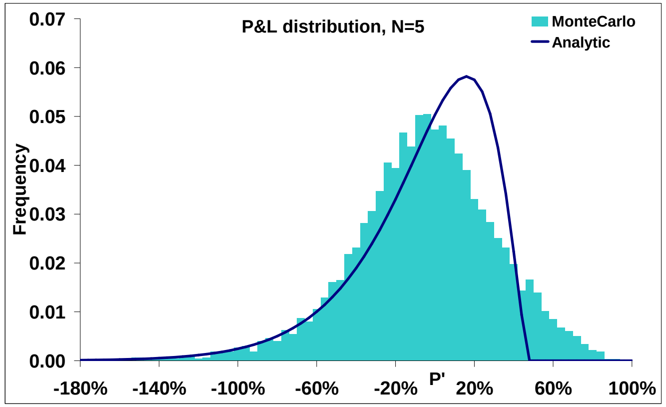


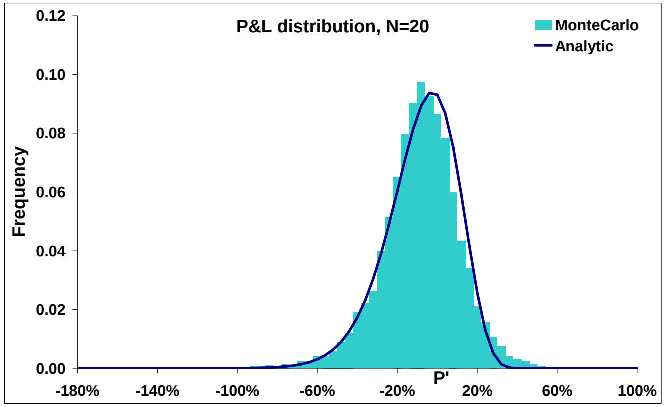


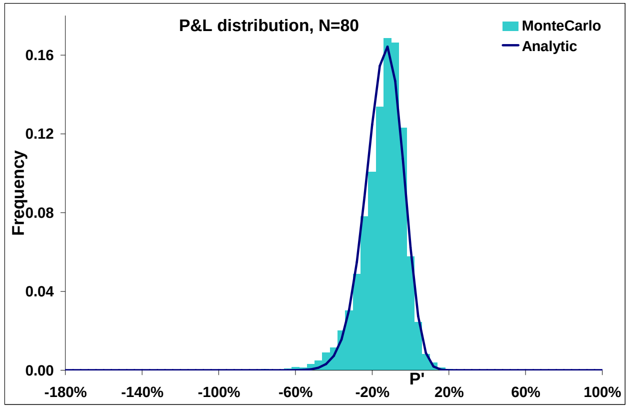


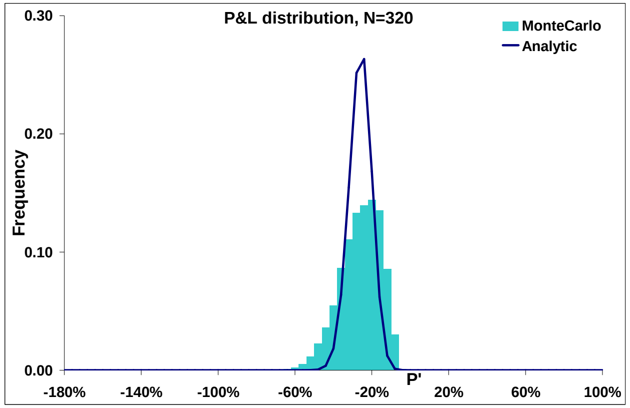


P&L Distribution in the DM with transaction costs

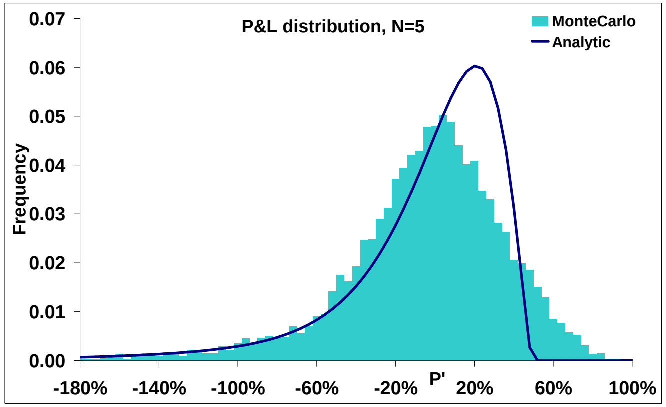


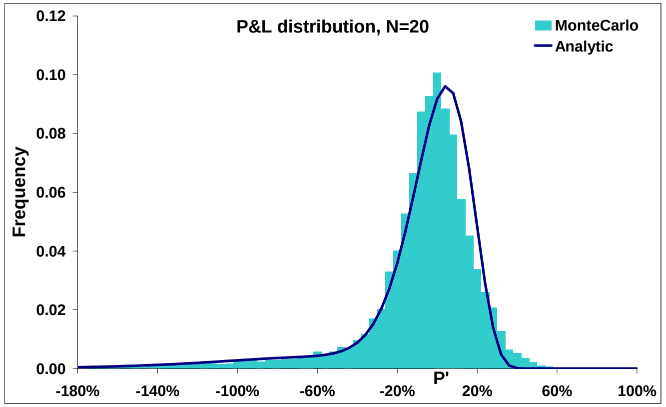


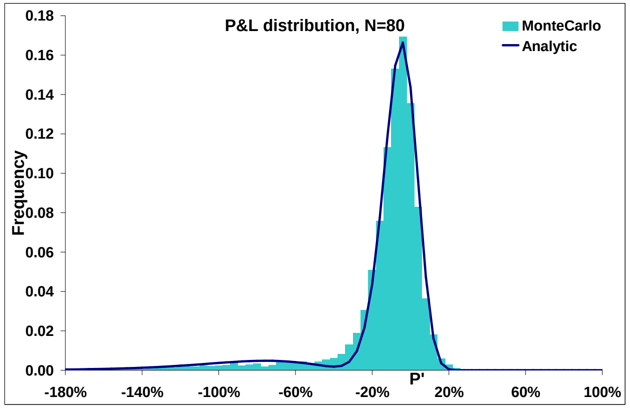


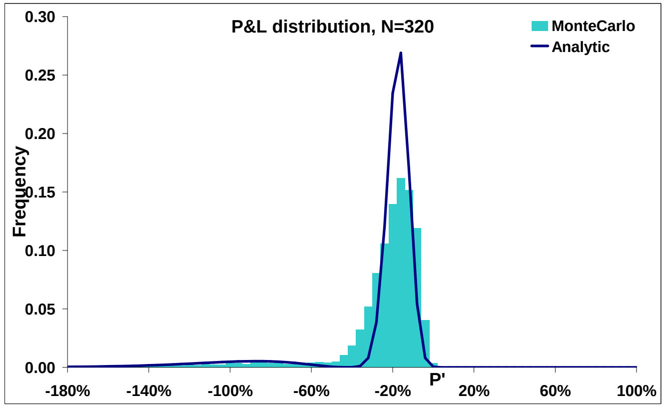


P&L Distribution in the JDM with transaction costs

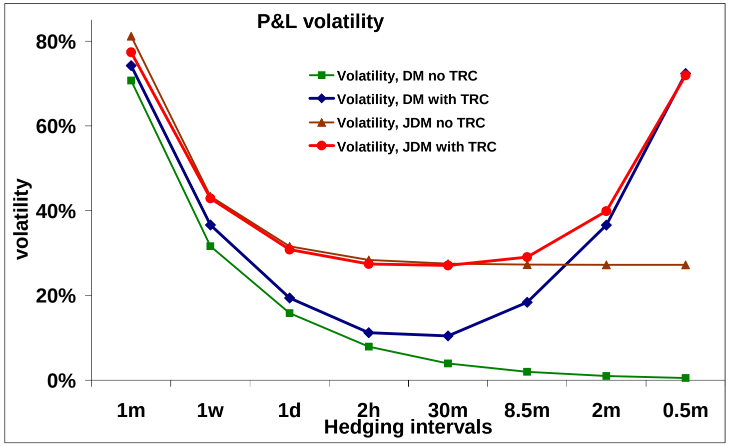


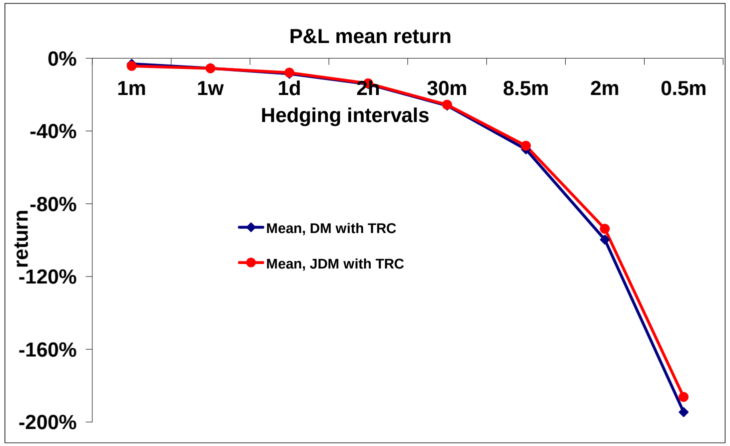






Illustration





Left: the P&L volatility under the DM (Volatility, DM no TRC) and JDM (Volatility, JDM no TRC) with no costs, and the P&L volatility under the DM (Volatility, DM with TRC) and JDM (Volatility, JDM with TRC) with proportional costs; Right: the mean of the P&L distribution under the DM (Mean, DM with TRC) and JDM (Mean, JDM with TRC) with proportional costs. The x-axes is the hedging frequency: 1m - the DHS is applied only $t = 0$, 1w - weekly, 1d - daily, 2h - every two hours, 30m - every thirty minutes, 8.5m - every eight minutes, 2m - every two minutes, 0.5m - every thirty seconds.

Jump Risk Hedge (Andersen-Andreasen (2000))
Define the jump risk $\mathbb{U}(t,S;T,K)$ under $\mathbb{Q}$ as follows:

$$\mathbb{U}(t,S;T,K) = \int_{-\infty}^{\infty} U(t,Se^J;T,K)\widetilde{w}(J)dJ - U(t,S;T,K)$$

Use series formula to obtain:

$$\mathbb{U}(t,S;T,K) = \sum_{m_1=0}^{\infty} ... \sum_{m_L=0}^{\infty} \mathcal{P}(m_1;p_1\overline{\widetilde{\lambda}}(T;t)(\mathbb{J}^{\mathbb{Q}}+1))...\mathcal{P}(m_L;p_L\overline{\widetilde{\lambda}}(T;t)(\mathbb{J}^{\mathbb{Q}}+1$$
$$\times \Big(\mathcal{C}^{(BSM)}((\mathbb{J}^{\mathbb{Q}}+1)S,K,1;\widetilde{\beta}(m_1+1,...,m_L+1),\widetilde{\alpha}(m_1,...,m_L),\overline{d}(T;t))$$
$$-\mathcal{C}^{(BSM)}(S,K,1;\widetilde{\beta}(m_1,...,m_L),\widetilde{\alpha}(m_1,...,m_L),\overline{d}(T;t))\Big)$$

The hedging portfolio $\Pi(t,S)$ to hedge $U^{(1)}(t,S)$:

$$\Pi(t,S) = S(t)\Delta^{(1)}(t) + U^{(2)}(t,S)\Delta^{(2)}(t) - U^{(1)}(t,S),$$

$\Delta^{(1)}(t)$ is the number of shares held in the underlying asset
$\Delta^{(2)}(t)$ is the number of units held in the complimentary option
here, either $K^{(1)} \neq K^{(2)}$ or $T^{(1)} \neq T^{(2)}$

Jump Risk Hedge

Require $\Pi(t,S)$ to be neutral to both delta and jump risks:

$$\Delta^{(2)}(t) = \frac{\mathbb{J}^{\mathbb{Q}} S(t) U_S^{(1)}(t,S) - \mathbb{U}^{(1)}(t,S)}{\mathbb{J}^{\mathbb{Q}} S(t) U_S^{(2)}(t,S) - \mathbb{U}^{(2)}(t,S)},$$

$$\Delta^{(1)}(t) = -\Delta^{(2)} U_S^{(2)}(t,S) + U_S^{(1)}(t,S)$$

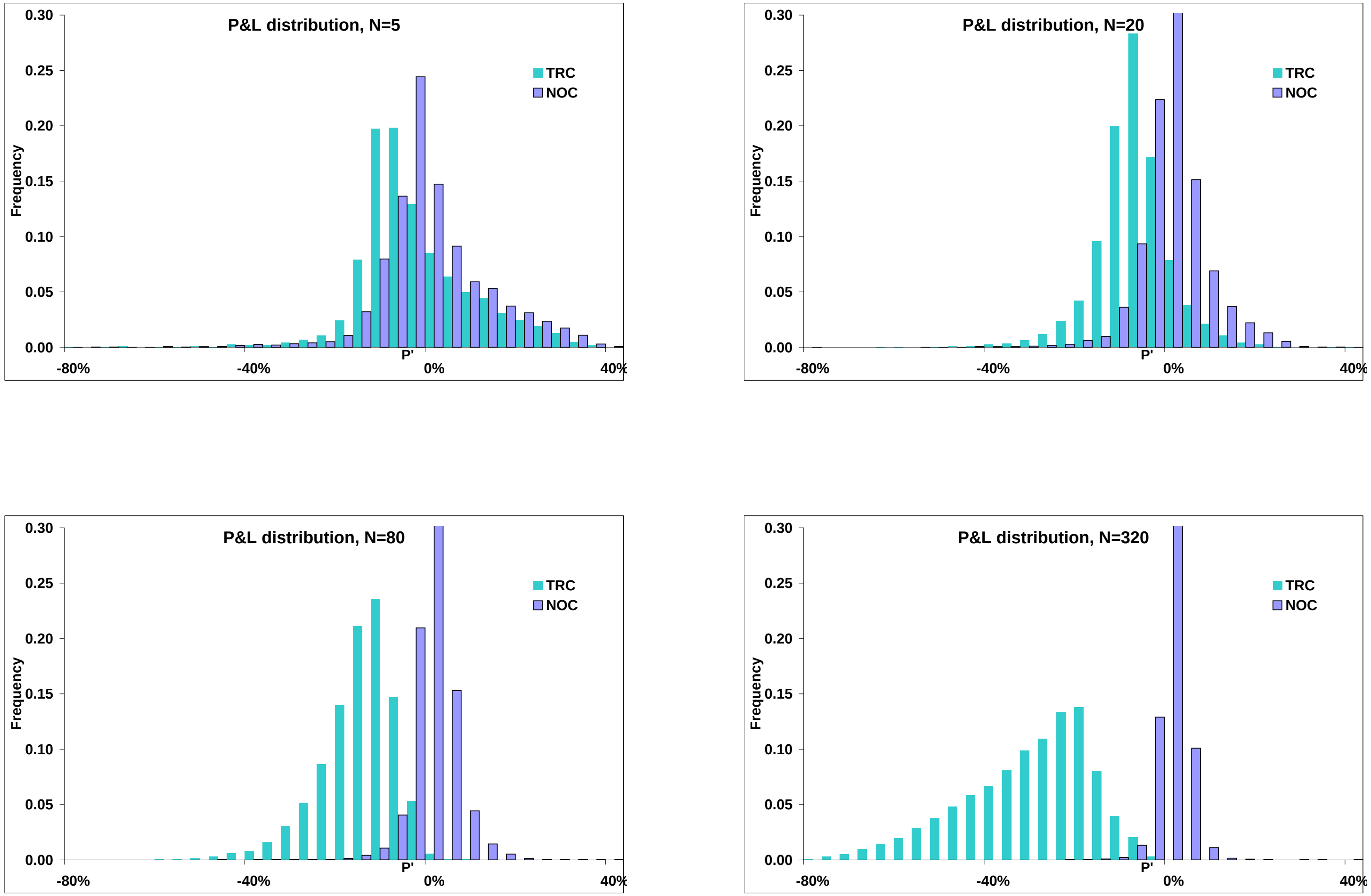
To get some intuition, we expand $\mathbb{U}$ in Taylor series around $J = 0$:

$$\Delta^{(2)}(t) \approx \frac{U_{SS}^{(1)}(t,S)}{U_{SS}^{(2)}(t,S)}$$

The hedging strategy is approximately gamma-neutral

For illustration use the parameters of the JDM with $K^{(1)} = K^{(2)} = 1$, $T^{(1)} = 0.08$, $K^{(2)} = 0.16$ without transaction costs and with transaction costs $\kappa = 0.002$

Jump Risk Hedge



Extensions. Variance swap. Consider a portfolio of delta-hedged positions, $\Pi^{(m)}(t_n, S)$, with weights $w^{(m)}$, in the same underlying:

$$\Upsilon(t_n, S) = \sum_{m=1}^{M} w^{(m)} \Pi^{(m)}(t_n, S),$$

where $\Pi^{(m)}(t_n, S) = S(t_n)\Delta^{(m)}(t_n, S) - U^{(m)}(t_n, S)$, $m = 1, ..., M$

The one-period P&L:

$$\delta\Upsilon(t_n, S) = \sum_{m=1}^{M} w^{(m)} \left(\widetilde{\vartheta}_n^{(m)} - \Sigma_n^2\right) \Gamma^{(m)}(t_{n-1}, S) = A - B\Sigma_n^2,$$

where $A = \sum_{m=1}^{M} w^{(m)} \widetilde{\vartheta}_n^{(m)} \Gamma^{(m)}(t_{n-1}, S)$, $B = \sum_{m=1}^{M} w^{(m)} \Gamma^{(m)}(t_{n-1}, S)$

For variance swap with unit notional, $A \equiv (AF/N^2)\sigma_k^2$, where σ_k^2 is variance strike, and $B \equiv AF/N$ (AF is annualization factor, N is sampling frequency)

Obtained expressions for the approximate P&L are exact for the P&L distribution of the variance swap (within our assumptions)

Conclusions

We have presented a quantitative approach to study the P&L distribution of the delta-hedging strategy assuming discrete trading and transaction costs under the diffusion model and jump-diffusion model

The opinions expressed in this presentation are those of the author alone and do not necessarily reflect the views and policies of Bank of America Merrill Lynch

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